Solution to Problem 13) We find the eigenvalues of the arbitrary 2×2 $ABCD$ matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{vmatrix} A-\lambda & B \\ C & D-\lambda \end{vmatrix} = 0 \rightarrow (A-\lambda)(D-\lambda) - BC = 0$$

$$\rightarrow \lambda^2 - (A+D)\lambda + (AD - BC) = 0 \rightarrow \lambda_{1,2} = ½(A+D) \pm ½\sqrt{(A-D)^2 + 4BC}$$

$$\rightarrow \lambda_1^2 + \lambda_2^2 = A^2 + D^2 + 2BC.$$

Next, we determine the right eigenvectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ of the matrix, as follows:

$$Ax_1 + Bx_2 = \lambda x_1 \quad \rightarrow \quad x_2 = \left(\frac{\lambda - A}{B}\right)x_1 = -\frac{(A-D) \mp \sqrt{(A-D)^2 + 4BC}}{2B}x_1.$$

In general, the two right eigen-vectors are *not* orthogonal to each other unless the matrix happens to be symmetric. Normalization condition:

$$x_1^2 + x_2^2 = x_1^2 + \frac{2(A-D)^2 + 4BC \mp 2(A-D)\sqrt{(A-D)^2+4BC}}{4B^2}x_1^2 = 1.$$

$$\rightarrow \quad x_1 = \frac{2B}{\sqrt{2(A-D)^2 + 4B^2 + 4BC \mp 2(A-D)\sqrt{(A-D)^2+4BC}}}.$$

$$\rightarrow \quad x_2 = -\frac{(A-D) \mp \sqrt{(A-D)^2+4BC}}{\sqrt{2(A-D)^2 + 4B^2 + 4BC \mp 2(A-D)\sqrt{(A-D)^2+4BC}}}.$$

The left eigen-values of the matrix are the same λ_1 and λ_2 as above. The left eigen-vectors $(y_1 \quad y_2)$ are found to be

$$(y_1 \quad y_2)\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \lambda(y_1 \quad y_2) \rightarrow Ay_1 + Cy_2 = \lambda y_1$$

$$\rightarrow \quad y_2 = \left(\frac{\lambda - A}{C}\right)y_1 = -\frac{(A-D) \mp \sqrt{(A-D)^2 + 4BC}}{2C}y_1.$$

Again, the two left eigen-vectors are *not* orthogonal to each other unless the matrix happens to be symmetric. Normalization condition:

$$y_1^2 + y_2^2 = y_1^2 + \frac{2(A-D)^2 + 4BC \mp 2(A-D)\sqrt{(A-D)^2+4BC}}{4C^2}y_1^2 = 1.$$

$$\rightarrow \quad y_1 = \frac{2C}{\sqrt{2(A-D)^2 + 4C^2 + 4BC \mp 2(A-D)\sqrt{(A-D)^2+4BC}}}.$$

$$\rightarrow \quad y_2 = -\frac{(A-D) \mp \sqrt{(A-D)^2+4BC}}{\sqrt{2(A-D)^2 + 4C^2 + 4BC \mp 2(A-D)\sqrt{(A-D)^2+4BC}}}.$$

The right and left eigen-vectors associated with *different* eigen-values are now seen to be orthogonal to each other. Normalization of the right and left eigen-vectors associated with the *same* eigenvalue requires the following condition to be satisfied:

$$x_1y_1 + x_2y_2 = x_1y_1 + \frac{2(A-D)^2 + 4BC \mp 2(A-D)\sqrt{(A-D)^2+4BC}}{4BC}x_1y_1 = 1$$

$$\rightarrow \quad x_1 = y_1 = \sqrt{\frac{2BC}{(A-D)^2 + 4BC \mp (A-D)\sqrt{(A-D)^2+4BC}}}.$$

The pair of right and left eigen-vectors associated with λ_1 (upper sign) and λ_2 (lower sign) are thus found to be

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\pm} = \sqrt{\frac{2BC}{(A-D)^2+4BC\mp(A-D)\sqrt{(A-D)^2+4BC}}} \begin{bmatrix} 1 \\ -\frac{(A-D)\mp\sqrt{(A-D)^2+4BC}}{2B} \end{bmatrix}.$$

$$(y_1 \quad y_2)^{\pm} = \sqrt{\frac{2BC}{(A-D)^2+4BC\mp(A-D)\sqrt{(A-D)^2+4BC}}} \begin{bmatrix} 1 & -\frac{(A-D)\mp\sqrt{(A-D)^2+4BC}}{2C} \end{bmatrix}.$$

The 2×2 matrices constructed from the right and left eigen-vectors are inverses of each other. They can be used to diagonalize the $ABCD$ matrix.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1^+ & x_1^- \\ x_2^+ & x_2^- \end{pmatrix} = \begin{pmatrix} x_1^+ & x_1^- \\ x_2^+ & x_2^- \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Reverse the multiplication order to verify that they are inverse matrices.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1^+ & x_1^- \\ x_2^+ & x_2^- \end{pmatrix} \begin{pmatrix} y_1^+ & y_2^+ \\ y_1^- & y_2^- \end{pmatrix} = \begin{pmatrix} x_1^+ & x_1^- \\ x_2^+ & x_2^- \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1^+ & y_2^+ \\ y_1^- & y_2^- \end{pmatrix}.$$

Consequently, the n^{th} power of the $ABCD$ matrix is found to be

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = \begin{pmatrix} x_1^+ & x_1^- \\ x_2^+ & x_2^- \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} y_1^+ & y_2^+ \\ y_1^- & y_2^- \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1^n x_1^+ y_1^+ + \lambda_2^n x_1^- y_1^- & \lambda_1^n x_1^+ y_2^+ + \lambda_2^n x_1^- y_2^- \\ \lambda_1^n x_2^+ y_1^+ + \lambda_2^n x_2^- y_1^- & \lambda_1^n x_2^+ y_2^+ + \lambda_2^n x_2^- y_2^- \end{pmatrix}.$$