Solution to Problem 11) A straightforward way to relate (x_0, y_0) to (x_0', y_0') is to use geometric constructs and show that $x_0' = x_0 \cos\theta + y_0 \sin\theta$ and $y_0' = y_0 \cos\theta - x_0 \sin\theta$. An alternative approach which leads to the same result is to represent the point (x_0, y_0) by the complex number $x_0 + \mathrm{i}y_0$ in the complex xy-plane. The rotation of the Cartesian xy-coordinate system through the angle θ then appears as an opposite rotation of the point $x_0 + \mathrm{i}y_0$ through the angle $-\theta$. Thus, in the new $x'y'$ coordinate system, the same point (x_0, y_0) may be represented by

$$x_0' + \mathrm{i}y_0' = (x_0 + \mathrm{i}y_0)\exp(-\mathrm{i}\theta) = (x_0 + \mathrm{i}y_0)(\cos\theta - \mathrm{i}\sin\theta)$$

$$= (x_0 \cos\theta + y_0 \sin\theta) + \mathrm{i}(y_0 \cos\theta - x_0 \sin\theta).$$

Therefore, rotation of the xy system through an angle θ requires multiplication of the coordinates (x_0, y_0) of an arbitrary point in the xy-plane by a rotation matrix, as follows:

$$\begin{pmatrix} x_0' \\ y_0' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

To find the eigen-values of the rotation matrix appearing in the above equation, we must solve its characteristic equation, namely,

$$\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\rightarrow \quad \lambda_\pm = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm \mathrm{i}\sin\theta = \exp(\pm\mathrm{i}\theta).$$

Subsequently, the eigen-vectors of the rotation matrix are determined as follows:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda_\pm \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \rightarrow \quad v_1\cos\theta + v_2\sin\theta = \lambda_\pm v_1$$

$$\rightarrow \; v_2 = (\lambda_\pm - \cos\theta)v_1/\sin\theta = (e^{\pm\mathrm{i}\theta} - \cos\theta)v_1/\sin\theta = \pm\mathrm{i}v_1 \; \rightarrow \; V_\pm = \begin{pmatrix} 1 \\ \pm\mathrm{i} \end{pmatrix} v_1.$$

Considering that v_1 is an arbitrary constant, we set it equal to 1, then form the matrix $\tilde{V}$ whose columns are the above eigen-vectors V_+ and V_-. The matrix $\tilde{V}$ and its inverse $\tilde{V}^{-1}$ are found to be

$$\tilde{V} = \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}; \qquad \tilde{V}^{-1} = \tfrac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}.$$

Diagonalization of the rotation matrix with the aid of the eigen-vector matrices $\tilde{V}$, $\tilde{V}^{-1}$, and the eigen-values matrix Λ, finally yields

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \tilde{V}\Lambda\tilde{V}^{-1} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\theta} & 0 \\ 0 & e^{-\mathrm{i}\theta} \end{pmatrix} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}.$$
