Solution to Problem 9) Consider the 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. If the eigen-values of the matrix happen to differ from one another, the matrix will be diagonalizable. So the first requirement for failure to diagonalize is the equality of the eigen-values. The characteristic equation for this matrix is

$$|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$
$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{21} = 0. \quad (1)$$

The two roots of the characteristic equation will coincide if and only if

$$(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = (a_{11} - a_{22})^2 + 4a_{12}a_{21} = 0. \quad (2)$$

The common eigen-value will then be $\lambda_1 = \lambda_2 = ½(a_{11} + a_{22})$, and the corresponding eigen-vectors must satisfy the following equations:

$$(A - \lambda I)V = \begin{pmatrix} ½a_{11} - ½a_{22} & a_{12} \\ a_{21} & ½a_{22} - ½a_{11} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3)$$

We recognize the following distinct situations:

i) $a_{11} = a_{22}$. In this case, in accordance with Eq.(2), either a_{12} or a_{21} or both must vanish. If $a_{12} = 0$ and $a_{21} \neq 0$, we will have a single eigen-vector $V_1 = V_2 = (0 \quad v_2)^T$, in which case the matrix A cannot be diagonalized. Similarly, If $a_{12} \neq 0$ and $a_{21} = 0$, the unique eigen-vector of A will be $V_1 = V_2 = (v_1 \quad 0)^T$, and, once again, A will not be diagonalizable. However, if $a_{12} = a_{21} = 0$, then any pair of values for v_1 and v_2 will be allowed. The matrix A will then have two linearly independent eigen-vectors, which enable its diagonalization.

ii) $a_{11} \neq a_{22}$. In this case, in accordance with Eq.(2), neither a_{12} nor a_{21} can vanish. Equation (3) will then admit only one solution, namely,

$$\frac{v_1}{v_2} = \frac{2a_{12}}{a_{22} - a_{11}} = \frac{a_{11} - a_{22}}{2a_{21}}. \quad (4)$$

Since the matrix A will have a single eigen-vector in this case, it cannot be diagonalized.

We conclude that a 2×2 matrix A is diagonalizable under only two circumstances: (a) When $(a_{11} - a_{22})^2 + 4a_{12}a_{21} \neq 0$, in which case the matrix will have a pair of distinct eigen-values, and two linearly independent eigen-vectors. (b) When $a_{11} = a_{22}$ and $a_{12} = a_{21} = 0$, in which case the matrix is already in a diagonal form.