Solution to Problem 6) The columns of the matrix are the vectors V_1, V_2, V_3 shown below:

$$V_1 = \begin{pmatrix} 1+\mathrm{i} \\ 2-3\mathrm{i} \\ 7 \end{pmatrix}; \qquad V_2 = \begin{pmatrix} 6 \\ 9-\mathrm{i} \\ 5+4\mathrm{i} \end{pmatrix}; \qquad V_3 = \begin{pmatrix} 3\mathrm{i} \\ 5-8\mathrm{i} \\ 2-\mathrm{i} \end{pmatrix}$$

Normalizing the vector V_1 requires that we compute its norm $\|V_1\|$, then divide all elements of V_1 by the norm, that is,

$$\|V_1\| = \sqrt{V_1^{*T} V_1} = \sqrt{|1+\mathrm{i}|^2 + |2-3\mathrm{i}|^2 + 7^2} = \sqrt{2+13+49} = 8.$$

$$W_1 = V_1/\|V_1\| = \tfrac{1}{8}\begin{pmatrix} 1+\mathrm{i} \\ 2-3\mathrm{i} \\ 7 \end{pmatrix}.$$

Next, we find the projections of V_2 and V_3 onto W_1, as follows:

$$W_1^{*T} V_2 = \tfrac{1}{8}(1-\mathrm{i} \quad 2+3\mathrm{i} \quad 7)\begin{pmatrix} 6 \\ 9-\mathrm{i} \\ 5+4\mathrm{i} \end{pmatrix} = (62+47\mathrm{i})/8.$$

$$W_1^{*T} V_3 = \tfrac{1}{8}(1-\mathrm{i} \quad 2+3\mathrm{i} \quad 7)\begin{pmatrix} 3\mathrm{i} \\ 5-8\mathrm{i} \\ 2-\mathrm{i} \end{pmatrix} = (51-5\mathrm{i})/8.$$

We now subtract from V_2 its projection onto W_1, then normalize the resulting vector, as follows:

$$V_2 - (W_1^{*T} V_2)W_1 = \begin{pmatrix} 6 \\ 9-\mathrm{i} \\ 5+4\mathrm{i} \end{pmatrix} - \tfrac{1}{64}(62+47\mathrm{i})\begin{pmatrix} 1+\mathrm{i} \\ 2-3\mathrm{i} \\ 7 \end{pmatrix} \cong \begin{pmatrix} 5.7656-1.7031\mathrm{i} \\ 4.8594+0.4375\mathrm{i} \\ -1.78125-1.1406\mathrm{i} \end{pmatrix}.$$

$$\|V_2 - (W_1^{*T} V_2)W_1\| \cong \sqrt{36.1427+23.8052+4.4738} \cong 8.0263.$$

$$W_2 = [V_2 - (W_1^{*T} V_2)W_1]/\|V_2 - (W_1^{*T} V_2)W_1\| = \begin{pmatrix} 0.7183-0.2122\mathrm{i} \\ 0.6054+0.0545\mathrm{i} \\ -0.2219-0.1421\mathrm{i} \end{pmatrix}.$$

In the next step, we compute the projection of V_3 onto W_2, namely,

$$W_2^{*T} V_3 = (0.7183+0.2122\mathrm{i} \quad 0.6054-0.0545\mathrm{i} \quad -0.2219+0.1421\mathrm{i})\begin{pmatrix} 3\mathrm{i} \\ 5-8\mathrm{i} \\ 2-\mathrm{i} \end{pmatrix}.$$

$$= 1.6527-2.4547\mathrm{i}.$$

Subtracting from V_3 its projections onto W_1 and W_2 now yields

$$V_3 - (W_1^{*T}V_3)W_1 - (W_2^{*T}V_3)W_2 = \begin{pmatrix} 3\text{i} \\ 5-8\text{i} \\ 2-\text{i} \end{pmatrix} - \frac{1}{64}(51-5\text{i})\begin{pmatrix} 1+\text{i} \\ 2-3\text{i} \\ 7 \end{pmatrix}$$

$$-(1.6527-2.4547\text{i})\begin{pmatrix} 0.7183-0.2122\text{i} \\ 0.6054+0.0545\text{i} \\ -0.2219-0.1421\text{i} \end{pmatrix}$$

$$= \begin{pmatrix} 3\text{i} \\ 5-8\text{i} \\ 2-\text{i} \end{pmatrix} - \begin{pmatrix} 0.875+0.71875\text{i} \\ 1.3594-2.5469\text{i} \\ 5.5781-0.5469\text{i} \end{pmatrix} - \begin{pmatrix} 0.6662-2.1139\text{i} \\ 1.1343-1.396\text{i} \\ -0.7155+0.3098\text{i} \end{pmatrix}$$

$$= \begin{pmatrix} -1.5412+4.3951\text{i} \\ 2.5063-4.0571\text{i} \\ -2.8626-0.7629\text{i} \end{pmatrix}.$$

It remains to compute the norm of the above vector, then normalize the vector by its norm, in order to arrive at W_3, that is,

$$\|V_3 - (W_1^{*T}V_3)W_1 - (W_2^{*T}V_3)W_2\| \cong \sqrt{21.6922+22.7416+8.7765} \cong 7.2945.$$

$$W_3 = \frac{V_3-(W_1^{*T}V_3)W_1-(W_2^{*T}V_3)W_2}{\|V_3-(W_1^{*T}V_3)W_1-(W_2^{*T}V_3)W_2\|} = \begin{pmatrix} -0.2113+0.6025\text{i} \\ 0.3436-0.5562\text{i} \\ -0.3924-0.1046\text{i} \end{pmatrix}.$$

The matrix formed by the orthonormalized column vectors W_1, W_2, W_3 may now be written down, as follows:

$$(W_1 \mid W_2 \mid W_3) = \begin{pmatrix} 0.125+0.125\text{i} & 0.7183-0.2122\text{i} & -0.2113+0.6025\text{i} \\ 0.250-0.375\text{i} & 0.6054+0.0545\text{i} & 0.3436-0.5562\text{i} \\ 0.875 & -0.2219-0.1421\text{i} & -0.3924-0.1046\text{i} \end{pmatrix}.$$

The conjugate transpose of the above matrix is readily seen to be

$$\begin{pmatrix} W_1^{*T} \\ W_2^{*T} \\ W_3^{*T} \end{pmatrix} = \begin{pmatrix} 0.125-0.125\text{i} & 0.250+0.375\text{i} & 0.875 \\ 0.7183+0.2122\text{i} & 0.6054-0.0545\text{i} & -0.2219+0.1421\text{i} \\ -0.2113-0.6025\text{i} & 0.3436+0.5562\text{i} & -0.3924+0.1046\text{i} \end{pmatrix}.$$

When the above 3×3 matrices are multiplied together, the result will be the identity matrix. The conjugate transpose of $(W_1 \mid W_2 \mid W_3)$ is thus seen to be its inverse as well.