Solution to Problem 5) a) In matrix form, the equations may be written as follows:

$$\begin{pmatrix} 2 & 5 & -6 \\ 7 & -4 & 9 \\ 5 & -6 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}.$$

Upon dividing the first row by 2, the equation becomes

$$\begin{pmatrix} 1 & 2.5 & -3 \\ 7 & -4 & 9 \\ 5 & -6 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 4 \\ 7 \end{pmatrix}.$$

Subtracting 7 times the first row from the second, and 5 times the first row from the third, we arrive at

$$\begin{pmatrix} 1 & 2.5 & -3 \\ 0 & -21.5 & 30 \\ 0 & -18.5 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -6.5 \\ -0.5 \end{pmatrix}.$$

Normalizing the second row by -21.5 yields

$$\begin{pmatrix} 1 & 2.5 & -3 \\ 0 & 1 & -60/43 \\ 0 & -18.5 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 13/43 \\ -0.5 \end{pmatrix}.$$

Upon adding 18.5 times the second row to the third row, we find

$$\begin{pmatrix} 1 & 2.5 & -3 \\ 0 & 1 & -60/43 \\ 0 & 0 & 7-(1110/43) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 13/43 \\ (481/86)-0.5 \end{pmatrix}.$$

We now start at the bottom row, and use back-substitution as we move up, to arrive at

$$x_3 = -0.2707,$$

$$x_2 - (60/43)x_3 = 13/43 \quad \rightarrow \quad x_2 = -0.0754,$$

$$x_1 + 2.5x_2 - 3x_3 = 1.5 \quad \rightarrow \quad x_1 = 0.8764.$$

b) The set of three linear equations in three unknowns is put in matrix form, as follows:

$$\begin{pmatrix} -1 & 4 & -5 \\ 3 & -12 & 13 \\ 2 & -8 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 12 \end{pmatrix}.$$

Normalizing the first row (i.e., multiplying it by -1) yields

$$\begin{pmatrix} 1 & -4 & 5 \\ 3 & -12 & 13 \\ 2 & -8 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 12 \end{pmatrix}.$$

To eliminate the 1$^{\text{st}}$ element of the 2$^{\text{nd}}$ row, subtract 3 times the 1$^{\text{st}}$ row from the 2$^{\text{nd}}$, and to eliminate the 1$^{\text{st}}$ element of the 3$^{\text{rd}}$ row, subtract twice the 1$^{\text{st}}$ row from the 3$^{\text{rd}}$.

$$\begin{pmatrix} 1 & -4 & 5 \\ 0 & 0 & -2 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix}.$$

Upon dividing the 2^{nd} row by -2, we find

$$\begin{pmatrix} 1 & -4 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0.5 \\ 6 \end{pmatrix}.$$

Subtracting 7 times the 2^{nd} row from the 3^{rd} row yields

$$\begin{pmatrix} 1 & -4 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3.0 \\ 0.5 \\ 2.5 \end{pmatrix}.$$

The last row does *not* provide any acceptable solutions, indicating that the original set of three equations in three unknowns is inconsistent and, therefore, unsolvable.

c) In matrix form, the set of 3 linear equations in 3 unknowns is written as follows:

$$\begin{pmatrix} 2 & -4 & 6 \\ 1 & 1 & 4 \\ 2 & 2 & 8 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}.$$

Normalization of the first row by 2 yields

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 4 \\ 2 & 2 & 8 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0.5 \\ 3 \\ 6 \end{pmatrix}.$$

Subtract the 1^{st} row from the 2^{nd}, and subtract twice the 1^{st} row from the 3^{rd} to find

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0.5 \\ 2.5 \\ 5 \end{pmatrix}.$$

Upon normalizing the 2^{nd} row by dividing it by 3, we arrive at

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & \tfrac{1}{3} \\ 0 & 6 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0.5 \\ \tfrac{5}{6} \\ 5 \end{pmatrix}.$$

We now subtract 6 times the 2^{nd} row from the third, and find

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & \tfrac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0.5 \\ \tfrac{5}{6} \\ 0 \end{pmatrix}.$$

The coefficients matrix is now in standard upper triangular form. The last row is identically satisfied irrespective of the values of α, β, and γ. consequently, this row does not contribute to the solution. The second row yields $\beta + \tfrac{1}{3}\gamma = \tfrac{5}{6}$. Therefore, $\gamma = 2.5 - 3\beta$ is an acceptable solution for γ (for any value of β). Finally, the first row yields $\alpha = 0.5 + 2\beta - 3\gamma = 11\beta - 7$. The original set of equations is thus seen to admit an infinite number of solutions, in which β is arbitrary, $\alpha = 11\beta - 7$, and $\gamma = 2.5 - 3\beta$.