Solution to Problem 7) The infinite-product formula of Sec.4.2, Eq.(53), is as follows:

$$\Gamma(z+1) = \lim_{N\to\infty} \frac{N!\,(N+1)^z}{(z+1)(z+2)(z+3)\cdots(z+N)}. \tag{1}$$

We thus have

$$\begin{aligned}
&\Gamma(z)\Gamma(z+⅓)\Gamma(z+⅔) \\
&\quad = \lim_{N\to\infty} \frac{(N!)^3(N+1)^{z-1}(N+1)^{z+⅓-1}(N+1)^{z+⅔-1}}{z(z+⅓)(z+⅔)\times(z+1)(z+1+⅓)(z+1+⅔)\times\cdots\times(z-1+N)(z+⅓-1+N)(z+⅔-1+N)} \\
&\quad = \lim_{N\to\infty} \frac{3^{3N}(N!)^3(N+1)^{3z-2}}{(3z)(3z+1)(3z+2)\times(3z+3)(3z+4)(3z+5)\times\cdots\times(3z-3+3N)(3z-2+3N)(3z-1+3N)} \\
&\quad = \lim_{N\to\infty} \frac{3^{3N}\big[(N!)^3/(3N)!\big](N+1)^{-1}[(N+1)/(3N+1)]^{3z-1}(3N)!\times(3N+1)^{3z-1}}{(3z-1+1)(3z-1+2)(3z-1+3)\cdots(3z-1+3N-2)(3z-1+3N-1)(3z-1+3N)}.
\end{aligned} \tag{2}$$

In the limit of large N, Stirling's asymptotic formula, $N! \sim \sqrt{2\pi N}(N/e)^N$, yields

$$\frac{(N!)^3}{(3N)!} \sim \frac{(2\pi N)^{3/2}(N/e)^{3N}}{\sqrt{2\pi\times 3N}\,(3N/e)^{3N}} = \frac{2\pi N}{\sqrt{3}\times 3^{3N}}. \tag{3}$$

We also invoke the fact that the term $[(N+1)/(3N+1)]^{3z-1}$ appearing in the numerator of Eq.(2) approaches $(⅓)^{3z-1}$ in the limit when $N\to\infty$. The end result is

$$\begin{aligned}
\Gamma(z)\Gamma(z+⅓)\Gamma(z+⅔) &= \lim_{N\to\infty} \frac{2\pi[N/(N+1)]\times 3^{½-3z}\times(3N)!\times(3N+1)^{3z-1}}{(3z-1+1)(3z-1+2)(3z-1+3)\cdots(3z-1+3N)} \\
&= 2\pi(3^{½-3z})\Gamma(3z).
\end{aligned} \tag{4}$$
