Solution to Problem 4) In the vicinity of the pole at $z = z_n = -n$, where $n = 0, 1, 2, \cdots$, the reflection formula yields

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) \qquad \rightarrow \qquad \Gamma(z) = \frac{\pi}{\Gamma(1-z)\sin(\pi z)}.$$

Expanding the singular term of the denominator in a Taylor series around $z = z_n$, we find

$$\Gamma\left(z_n + \varepsilon e^{\mathrm{i}\theta}\right) = \frac{\pi}{\Gamma(1+n-\varepsilon e^{\mathrm{i}\theta})\sin\left[\pi(-n+\varepsilon e^{\mathrm{i}\theta})\right]} \cong \frac{\pi}{n! \times \left[\underbrace{\sin(-n\pi)}_{\text{zero}} + \pi \underbrace{\cos(-n\pi)}_{(-1)^n}\varepsilon e^{\mathrm{i}\theta}\right]} = \frac{(-1)^n}{n!\,(\varepsilon e^{\mathrm{i}\theta})}.$$

Integration around a small circle of radius ε involves multiplication of the above function by $\mathrm{d}z = \mathrm{i}\varepsilon e^{\mathrm{i}\theta}$, followed by integration over θ from zero to 2π, which yields $2\pi\mathrm{i}$ times $(-1)^n/n!$. The residue of $\Gamma(z)$ at the simple pole z_n is thus given by $(-1)^n/n!$.