Solution to Problem 3) Let $f(t) = e^{-t}t^z$ and $g'(t) = t^{-1}$. Integration by parts yields

$$\int_0^\infty f(t)g'(t)\mathrm{d}t = f(t)g(t)|_0^\infty - \int_0^\infty f'(t)g(t)\mathrm{d}t.$$

Given that $g(t) = \ln t$, it is seen that $\lim_{t\to 0} f(t)\,g(t) = \lim_{t\to 0}(e^{-t}t^z \ln t)$ is zero when $\mathrm{Re}(z) > 0$, and that $\lim_{t\to\infty}(e^{-t}t^z \ln t)$ is also zero, irrespective of the value of z. Consequently,

$$\Gamma(z) = -\int_0^\infty f'(t)g(t)\mathrm{d}t = -\int_0^\infty (e^{-t}t^z)' \ln t\, \mathrm{d}t$$

$$= -\int_0^\infty (-e^{-t}t^z + e^{-t}zt^{z-1}) \ln t\, \mathrm{d}t = \int_0^\infty e^{-t}(t-z)t^{z-1} \ln t\, \mathrm{d}t\,, \quad \mathrm{Re}(z) > 0.$$