Problem 20) Given that the partial differential could be written as $[f(x,t)\hat{\boldsymbol{x}}+\hat{\boldsymbol{t}}]\cdot\boldsymbol{\nabla}f(x,t)=0$, the function $f(x,t)$ remains constant as one moves along the vector $\boldsymbol{v}=f(x,t)\hat{\boldsymbol{x}}+\hat{\boldsymbol{t}}$ within the xt-plane. The initial condition is specified along the x-axis at $t=0$, where a typical set of $\boldsymbol{v}$ vectors originating on the x-axis are shown in Fig.1(a). The function remains constant as one moves along individual $\boldsymbol{v}$ vectors, and, consequently, the magnitude and direction of each $\boldsymbol{v}$ vector remain intact as one moves to larger values of t, away from the x-axis. The process continues so long as the $\boldsymbol{v}$ vectors do not cross each other. Once a pair of these vectors cross, the solution breaks down. It is thus clear that, up until the breakdown, the solution may be written as an implicit function $f(x,t)=f_0(x-ft)$. Note that the same $f(x,t)$ appears on both sides of the equation. Depicted in Fig.1(b) is a typical initial distribution $f_0(x)$ as it evolves in time and approaches $f(x,t_0)$ at a later time $t=t_0$. The breakdown occurs if and when the function acquires a vertical tangent, that is, $\partial_x f(x,t)\to\infty$ for some value(s) of x.

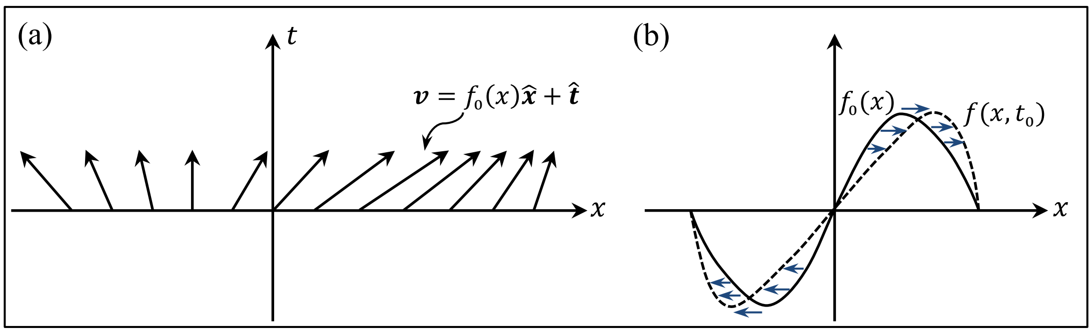

Fig.1. (a) At $t=0$, the solution $f(x,t)$ of the PDE moves along the characteristic base curves in the directions specified by the local $\boldsymbol{v}$ vectors. (b) The initial profile $f_0(x)$ of the function $f(x,t)$ is deformed as time progresses.

Case i) $f_0(x)=-x$. Here, $f(x,t)=-[x-f(x,t)t]$, which is readily solved to yield the explicit solution $f(x,t)=x/(t-1)$. Clearly, the slope of the function starts at -1 at $t=0$, then increases with time until breakdown occurs at $t=1$.

Case ii) $f_0(x)=\sin(x)$. Here, $f(x,t)=\sin(x-ft)$. The slope of the function at point x at time t may thus be computed as follows:

$$\partial_x f=(1-t\partial_x f)\cos(x-ft)\quad\rightarrow\quad\partial_x f(x,t)=\frac{\cos(x-ft)}{1+t\cos(x-ft)}.$$

The earliest time at which the slope of $f(x,t)$ becomes infinite is $t=1$, at locations where the function $\cos(x-ft)$ equals -1, namely, $x=\pm\pi,\pm3\pi,\pm5\pi,\cdots$.

Case iii) $f_0(x)=1-x^2$. Here $f(x,t)=1-(x-ft)^2$. This quadratic equation in $f(x,t)$ is easily solved to yield

$$f(x,t)=\frac{(2xt-1)\pm\sqrt{1+4t(t-x)}}{2t^2}.$$ ← use $\sqrt{1+\varepsilon}\cong1+½\varepsilon-⅛\varepsilon^2$.

The above solution must approach $f_0(x)=1-x^2$ in the limit when $t\to0^+$, which dictates the choice of $+$ sign in the numerator. Subsequently, the local slope of the function is found to be

$$\partial_x f=\frac{1}{t}\left[1-\frac{1}{\sqrt{1+4t(t-x)}}\right].$$ ← use $(1+\varepsilon)^{-½}\cong1-½\varepsilon+⅜\varepsilon^2$.

The apparent singularity of $\partial_x f(x,t)$ at $t=0$ disappears when the radical is properly expanded in a Taylor series. The true singularity, however, occurs when the radical itself approaches zero, that is, at $x = t + ¼t^{-1}$. Figure 2(a) shows the initial stages of the evolution of $f_0(x)$, immediately after $t=0$. The plot of the function $x = t + ¼t^{-1}$ in Fig.2(b) reveals that breakdown occurs at different times (before and up to $t = ½$) in regions where $x \geq 1$.

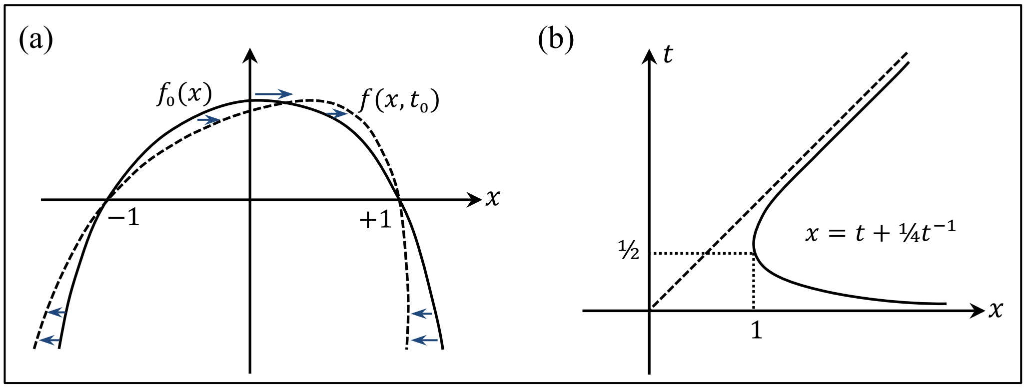

Fig.2. (a) Evolution of $f(x,t)$ away from $f_0(x)$ during a short time interval following $t=0$. (b) Plot of the function $x = t + ¼t^{-1}$, on which the slope of $f(x,t)$ goes to infinity. Only the lower-half of the plot is relevant, as it pertains to the earliest time at which breakdown occurs.