Problem 18) Separation of Variables: $T(r,z) = f(r)g(z)$.

Substitution into the differential equation yields:

$$\frac{\partial^2 T(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,z)}{\partial r} + \frac{\partial^2 T(r,z)}{\partial z^2} = f''(r)g(z) + \frac{1}{r}f'(r)g(z) + f(r)g''(z) = 0$$

$$\Rightarrow \frac{f''(r)}{f(r)} + \frac{1}{r}\frac{f'(r)}{f(r)} + \frac{g''(z)}{g(z)} = 0 \Rightarrow$$

$$\frac{g''(z)}{g(z)} = +c^2$$ (positive separation constant, for reasons to become clear shortly.)

$\Rightarrow g(z) = Ae^{cz} + Be^{-cz}$. The positive exponential is not acceptable, because its slope does not vanish as $z \to \infty$. The only acceptable solution, therefore, is $g(z) = Be^{-cz}$.

$$\frac{f''(r)}{f(r)} + \frac{1}{r}\frac{f'(r)}{f(r)} + c^2 = 0 \Rightarrow r^2 f''(r) + r f'(r) + c^2 r^2 f(r) = 0$$

This is the Bessel equation of order zero. The general solution to this equation is $f(r) = DJ_0(cr) + EY_0(cr)$. However, $Y_0(cr)$ is divergent at the center of the cylinder $(r=0)$ and, therefore, is unacceptable. We must set $E=0$ to eliminate the Bessel function of the second kind. The remaining solution $f(r) = DJ_0(cr)$ is acceptable for all values of $c>0$, because in the limit $r = R \to \infty$, the derivative of $J_0(cr)$ is zero, thereby ensuring no heat-loss from the cylindrical surface in the limit of large cylinder (i.e., $R \to \infty$). The separable solution is thus given by $f(r)g(z) = BDe^{-cz}J_0(cr)$. We can combine the coefficients BD into a single coefficient A, where A is a function of the separation parameter c. The general solution may now be written as a superposition integral over all values of c from zero to infinity. For aesthetic reasons, we replace c by a different variable, η, to obtain:

$$T(r,z) = \int_0^\infty A(\eta)\, e^{-\eta z} J_0(\eta r)\, d\eta .$$

It remains to match the boundary condition at $z=0$, where $T(r,0)=T_0(r)$.

We will have: $T(r,z=0) = T_0(r) = \int_0^\infty A(\eta) J_0(\eta r)\, d\eta$. From this equation one can determine the function $A(\eta)$ in terms of $T_0(r)$. Substitution in the general solution for $T(r,z)$ then yields the complete solution for the steady-state temperature distribution.

Digression: Although we have not discussed the Hankel Transform in this course, the determination of $A(\eta)$ is best done using the Hankel Transform of $T_0(r)$. By definition the Hankel Transform of $T_0(r)$ [denoted by $\hat{T}_0(\eta)$] is given by:

$$\hat{T}_0(\eta) = \int_0^\infty r\, T_0(r) J_0(\eta r)\, dr \iff T_0(r) = \int_0^\infty \eta\, \hat{T}_0(\eta) J_0(r\eta)\, d\eta$$

inverse Hankel Transform

Comparison with the equation $T_0(r) = \int_0^\infty A(\eta) J_0(\eta r)\, d\eta$ shows that $\eta \hat{T}_0(\eta) = A(\eta)$.