Problem 16) We saw in Sec.9 that, for a solid disk of radius R, the method of separation of variables leads to the following general solution:

$$T(r,\varphi) = B_0 + \sum_{m=1}^{\infty} r^m [A_m \sin(m\varphi) + B_m \cos(m\varphi)]. \tag{1}$$

The pre-specified temperature at the boundary can be expanded into a Fourier series, as follows:

$$f(\varphi) = b_0 + \sum_{m=1}^{\infty} [a_m \sin(m\varphi) + b_m \cos(m\varphi)]. \tag{2}$$

Here

$$b_0 = (1/2\pi) \int_0^{2\pi} f(\theta)\mathrm{d}\theta. \tag{3a}$$

$$a_m = (1/\pi) \int_0^{2\pi} f(\theta) \sin(m\theta)\, \mathrm{d}\theta. \tag{3b}$$

$$b_m = (1/\pi) \int_0^{2\pi} f(\theta) \cos(m\theta)\, \mathrm{d}\theta. \tag{3c}$$

Matching the boundary conditions at $r = R$, we find: $B_0 = b_0$, $A_m = a_m/R^m$, $B_m = b_m/R^m$. Substitution into Eq.(1) now yields

$$T(r,\varphi) = (1/2\pi) \int_0^{2\pi} f(\theta)\mathrm{d}\theta$$

$\sin a \sin b + \cos a \cos b = \cos(a-b)$

$$+(1/\pi) \sum_{m=1}^{\infty} (r/R)^m \int_0^{2\pi} f(\theta)[\sin(m\theta)\sin(m\varphi) + \cos(m\theta)\cos(m\varphi)]\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} f(\theta)\mathrm{d}\theta + 2\sum_{m=1}^{\infty} (r/R)^m \int_0^{2\pi} f(\theta)\cos[m(\varphi-\theta)]\,\mathrm{d}\theta\right\}$$

$\cos x = ½(e^{\mathrm{i}x} + e^{-\mathrm{i}x})$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} f(\theta)\mathrm{d}\theta + \sum_{m=1}^{\infty} \int_0^{2\pi} (r/R)^m \left[e^{\mathrm{i}m(\varphi-\theta)} + e^{-\mathrm{i}m(\varphi-\theta)}\right] f(\theta)\mathrm{d}\theta\right\}$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} f(\theta)\mathrm{d}\theta + \int_0^{2\pi} \sum_{m=1}^{\infty} \{[(r/R)e^{\mathrm{i}(\varphi-\theta)}]^m + [(r/R)e^{-\mathrm{i}(\varphi-\theta)}]^m\} f(\theta)\,\mathrm{d}\theta\right\}$$

$\sum_{m=1}^{\infty} x^m = x/(1-x)$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[1 + \frac{(r/R)e^{\mathrm{i}(\varphi-\theta)}}{1-(r/R)e^{\mathrm{i}(\varphi-\theta)}} + \frac{(r/R)e^{-\mathrm{i}(\varphi-\theta)}}{1-(r/R)e^{-\mathrm{i}(\varphi-\theta)}}\right] f(\theta)\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[1 + \frac{re^{\mathrm{i}(\varphi-\theta)}}{R-re^{\mathrm{i}(\varphi-\theta)}} + \frac{re^{-\mathrm{i}(\varphi-\theta)}}{R-re^{-\mathrm{i}(\varphi-\theta)}}\right] f(\theta)\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\theta)+r^2} f(\theta)\mathrm{d}\theta. \tag{4}$$
