Problem 15) Heat diffusion equation: $D\nabla^2 T(r,\phi,z,t) = \frac{\partial T(r,\phi,z,t)}{\partial t} \Rightarrow$

$$D\left\{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right\} = \frac{\partial T}{\partial t}$$

Because of circular symmetry, we may ignore the ϕ-dependence of T.

Separation of variables then yields $T(r,\phi,z,t) = g(r)\,h(z)\,p(t) \Rightarrow$

$$D\left\{g''(r)h(z)p(t) + \frac{1}{r}g'(r)h(z)p(t) + g(r)h''(z)p(t)\right\} = g(r)h(z)p'(t) \Rightarrow$$

$$D\left\{\frac{g''(r)}{g(r)} + \frac{1}{r}\frac{g'(r)}{g(r)} + \frac{h''(z)}{h(z)}\right\} = \frac{p'(t)}{p(t)}$$

$$\frac{g''(r)}{g(r)} + \frac{1}{r}\frac{g'(r)}{g(r)} = -k^2 \text{ (negative constant)} \Rightarrow r^2 g''(r) + r g'(r) + k^2 r^2 g(r) = 0$$

$\Rightarrow g(r) = J_0(kr) \leftarrow$ Bessel function of 1st kind, 0th order.

On the side walls of the cylinder $J_0(kr) = J_0(kR) = 0 \Rightarrow kR = r_{0n} \leftarrow n^{th}$ zero of $J_0(r)$

$$\Rightarrow g(r) = J_0\left(\frac{r_{0n}}{R}r\right) \text{ and } k = \frac{r_{0n}}{R}.$$

$$\frac{h''(z)}{h(z)} = -\alpha \text{ (negative constant)} \Rightarrow h(z) = \cos\left[\frac{(2m+1)\pi}{2L}z\right];\ m = 0, 1, 2, 3, \ldots$$

$\Rightarrow \alpha = \left[\frac{(2m+1)\pi}{2L}\right]^2$. This choice of α ensures that $h(z=-L) = 0$ and

also $\left.\frac{\partial h(z)}{\partial z}\right|_{z=0} = 0$, so that no heat escapes for the top surface.

$$\frac{p'(t)}{p(t)} = -D\left\{\left(\frac{r_{0n}}{R}\right)^2 + \left[\frac{(2m+1)\pi}{2L}\right]^2\right\} \Rightarrow p(t) = e^{-\left\{\left(\frac{r_{0n}}{R}\right)^2 + \left[\frac{(2m+1)\pi}{2L}\right]^2\right\}Dt}$$

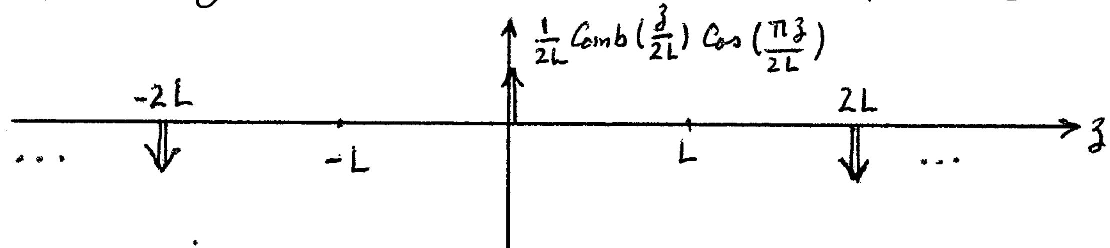
$$T(r,\phi,z,t) = T_0 + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{nm}\, J_0\left(\frac{r_{0n}}{R}r\right)\cos\left[\frac{(2m+1)\pi}{2L}z\right] e^{-\left\{\left(\frac{r_{0n}}{R}\right)^2 + \left[\frac{(2m+1)\pi}{2L}\right]^2\right\}Dt}$$

The coefficients A_{nm} of the various modes are determined by matching the above solution to the initial condition at $t=0$, namely,

$$T(r,\phi,z,0) - T_0 = f(r)\,\delta(z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{nm} J_0\left(\frac{r_{0n}}{R} r\right) \cos\left[\frac{(2m+1)\pi}{2L} z\right]$$

This requires expanding $f(r)$ into a series of Bessel functions $J_0\left(\frac{r_{0n}}{R} r\right)$, where r_{0n} are the various zeros of $J_0(r)$, using standard methods of expansion.

The less straightforward problem is the expansion of $\delta(z)$ over the interval $-L \leq z \leq 0$ into a series containing only $\cos\left[\frac{(2m+1)\pi}{2L} z\right]$. We extend $\delta(z)$ over the interval $(-2L, 2L)$, such that it becomes an even function of z with respect to $z=0$, but an odd function with respect to $z=-L$, as follows:



The above function has the desired symmetry, and can be expanded in a series containing only $\cos\left[\frac{(2m+1)\pi}{2L} z\right]$ terms, as follows:

$$\mathcal{F}\left\{\frac{1}{2L}\mathrm{Comb}\left(\frac{z}{2L}\right)\cos\left(\frac{\pi z}{2L}\right)\right\} = \mathrm{Comb}(2Ls) * \left\{\frac{1}{2}\delta\left(s-\frac{1}{4L}\right) + \frac{1}{2}\delta\left(s+\frac{1}{4L}\right)\right\}$$

$$= \frac{1}{2}\left[\sum_{m=-\infty}^{\infty}\delta(2Ls-m)\right] * \left[\delta\left(s-\frac{1}{4L}\right)+\delta\left(s+\frac{1}{4L}\right)\right] = \frac{1}{4L}\left[\sum_{m=-\infty}^{\infty}\delta\left(s-\frac{m}{2L}\right)\right] * \left[\delta\left(s-\frac{1}{4L}\right)+\delta\left(s+\frac{1}{4L}\right)\right]$$

$$= \frac{1}{4L}\sum_{m=-\infty}^{\infty}\left[\delta\left(s-\frac{m+1/2}{2L}\right)+\delta\left(s-\frac{m-1/2}{2L}\right)\right] = \frac{1}{2L}\sum_{m=-\infty}^{\infty}\delta\left(s-\frac{2m+1}{4L}\right)$$

Therefore, $\frac{1}{2L}\mathrm{Comb}\left(\frac{z}{2L}\right)\cos\left(\frac{\pi z}{2L}\right) = \mathcal{F}^{-1}\left\{\frac{1}{2L}\sum_{m=-\infty}^{\infty}\delta\left(s-\frac{2m+1}{4L}\right)\right\} = \frac{1}{2L}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\left(s-\frac{2m+1}{4L}\right)e^{i2\pi s z}\,ds$

$$= \frac{1}{L}\sum_{m=0}^{\infty}\cos\left[\frac{(2m+1)\pi}{2L} z\right].$$

In the interval $-L \leq z \leq 0$ this series reproduces the initial temperature profile along the z-axis