Problem 14) Separation of Variables $T(x,y) = f(x)g(y) \Rightarrow$

$$f''(x)g(y) + f(x)g''(y) = 0 \Rightarrow \frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} = -c^2 \leftarrow \text{negative Constant}$$

The reasons for the choice of the separation variable as a negative parameter will become clear shortly.

$$-\frac{g''(y)}{g(y)} = -c^2 \Rightarrow g''(y) - c^2 g(y) = 0 \Rightarrow g(y) = Ae^{cy} + Be^{-cy}.$$ The solution with positive exponent, e^{cy}, is unacceptable because, at $y=\infty$ it will yield $\frac{\partial T(x,y)}{\partial y} \neq 0$, which means that heat is escaping from the upper boundary. We must therefore, set $A=0$ and accept $g(y) = Be^{-cy}$ as the only solution that matches the boundary condition as $y \to \infty$.

$$\frac{f''(x)}{f(x)} = -c^2 \Rightarrow f''(x) + c^2 f(x) = 0 \Rightarrow f(x) = D\cos(cx) + E\sin(cx).$$ Since the left-edge of the strip is insulated, we must have $f'(x)\big|_{x=0} = 0 \Rightarrow E = 0$. Since the right-edge of the strip is also insulated, we must have $f'(x)\big|_{x=L} = 0$

$\Rightarrow -Dc\sin(cL) = 0 \Rightarrow cL = m\pi, \; m = 0, 1, 2, 3, \dots \Rightarrow c = \frac{m\pi}{L}.$

The separable solution is thus given by $f(x)g(y) = BD\exp(-\frac{m\pi y}{L})\cos(\frac{m\pi x}{L})$. We now combine the arbitrary coefficients B and D into a single coefficient A_m, which depends on m, then write the general solution to the problem as a superposition of all separable solutions, that is,

$$T(x,y) = \sum_{m=0}^{\infty} A_m \exp\left(\frac{-m\pi y}{L}\right)\cos\left(\frac{m\pi x}{L}\right).$$

Note that the solution corresponding to $c=0$ is already present in the above expression as $m=0$. When solving $\frac{g''(y)}{g(y)}=0$, we find $g(y)=Ay+B$; again the solution Ay is unacceptable, because its derivation does not go to zero as $y\to\infty$. The remaining solution is thus $g(y)=B$, which is the same as $g(y)=Be^{-cy}$ when $c=0$.

Next, we must match the boundary condition at $y=0$, namely, $T(x,y=0)=T_0(x)=\sum_{m=0}^{\infty}A_m\cos\left(\frac{m\pi x}{L}\right)$. The right-hand side is an even function of x with a period of $2L$ along the x-axis. The left-hand-side, however, is only defined on the interval $(0,L)$. We thus extend $T_0(x)$ to the negative interval $(-L,0)$, defining the extension as $T_0(-x)=T_0(x)$, so that the extended function represents a single-period of an even function with a periodicity of $2L$. The coefficients A_m are thus obtained as follows:

$$\int_{-L}^{L}T_0(x)\,dx=2LA_0 \Rightarrow \int_{-L}^{0}T_0(-x)\,dx+\int_0^L T_0(x)\,dx=2\int_0^L T_0(x)\,dx=2LA_0 \Rightarrow$$

$$A_0=\frac{1}{L}\int_0^L T_0(x)\,dx.$$

$$\int_{-L}^{L}T_0(x)\cos\left(\frac{n\pi x}{L}\right)dx=\sum_{m=0}^{\infty}A_m\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx \Rightarrow$$

$$\int_{-L}^{0}T_0(-x)\cos\left(\frac{n\pi x}{L}\right)dx+\int_0^L T_0(x)\cos\left(\frac{n\pi x}{L}\right)dx=\sum_{m=0}^{\infty}\frac{1}{2}A_m\left\{\cancelto{0}{\int_{-L}^{L}\cos\left[\frac{(m+n)\pi x}{L}\right]dx}+\underbrace{\int_{-L}^{L}\cos\left[\frac{(m-n)\pi x}{L}\right]dx}_{\substack{0 \text{ when } m\neq n\\ 2L \text{ when } m=n}}\right\}$$

$$\Rightarrow 2\int_0^L T_0(x)\cos\left(\frac{n\pi x}{L}\right)dx=LA_n \Rightarrow A_n=\frac{2}{L}\int_0^L T_0(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

Note that the heat flows in and out of the strip at the boundary $y=0$. However, the integral of the heat-flow-rate along the x-axis, from $x=0$ to $x=L$, is exactly zero.