Problem 13) The heat diffusion equation in 3-dimensional Cartesian space is written as

$$D\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)T(x, y, z, t) = \partial_t T(x, y, z, t). \tag{1}$$

In the steady state, $\partial_t T = 0$, and the separation of variables $T(x, y, z) = f(x)g(y)h(z)$ yields

$$f''(x)g(y)h(z) + f(x)g''(y)h(z) + f(x)g(y)h''(z) = 0. \tag{2}$$

Upon dividing Eq.(2) by $f(x)g(y)h(z)$, we arrive at

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} + \frac{h''(z)}{h(z)} = 0. \tag{3}$$

The individual terms of the above equation must be constants. Keeping in mind the boundary conditions, we equate the first term in Eq.(3) to $-c_1^2$, and the second term to $-c_2^2$ (i.e., two arbitrary but negative constants). This yields $f(x) = A\sin(c_1 x) + B\cos(c_1 x)$. The boundary conditions along the x-axis then require that $B = 0$ and $c_1 = m\pi/L_x$, where $m = 1, 2, 3, \cdots$ could be any positive integer. Similarly, $g(y) = A'\sin(c_2 y) + B'\cos(c_2 y)$, which, upon enforcing the boundary conditions along the y-direction, yields $B' = 0$ and $c_2 = n\pi/L_y$, where $n = 1, 2, 3, \cdots$ is another arbitrary positive integer. (Note that the integers m and n are completely independent of each other.) The last term in Eq.(3) must now be equated to $c_1^2 + c_2^2$, which leads to $h(z) = A''\sinh\left(\sqrt{c_1^2 + c_2^2}\, z\right) + B''\cosh\left(\sqrt{c_1^2 + c_2^2}\, z\right)$. The boundary condition in the $z = 0$ plane requires that B'' be zero. The general solution to Eq.(1) is thus found to be

$$T(x, y, z) = T_0 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin(m\pi x/L_x)\sin(n\pi y/L_y)\sinh\left[\pi\sqrt{(m/L_x)^2 + (n/L_y)^2}\,z\right]. \tag{4}$$

The unknown coefficients A_{mn} must be obtained by matching the boundary condition at the top facet, $z = L_z$. We thus require that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sinh\left[\pi\sqrt{(mL_z/L_x)^2 + (nL_z/L_y)^2}\right]\sin(m\pi x/L_x)\sin(n\pi y/L_y) = T_1(x, y). \tag{5}$$

Note that $\sinh(\cdots)$ appearing in Eq.(5) is just a constant. The boundary temperature $T_1(x, y)$ should be expanded into a 2-dimensional Fourier sine series over the $2L_x \times 2L_y$ rectangular region shown on the right. To this end, both sides of Eq.(5) are multiplied by $\sin(m'\pi x/L_x)\sin(n'\pi y/L_y)$, then integrated over the area of the rectangle. The only non-zero integral will then correspond to $m' = m$ and $n' = n$, thus yielding the value of the $A_{m'n'}$ coefficient.