

Problem 11)

a) $f(x, t) = \int_{-\infty}^{\infty} F(s, t) \exp(i2\pi sx) ds \rightarrow \frac{\partial^2 f(x, t)}{\partial x^2} = \int_{-\infty}^{\infty} (i2\pi s)^2 F(s, t) \exp(i2\pi sx) ds.$

Thus the Fourier transform of $\partial^2 f(x, t)/\partial x^2$ is $-(2\pi s)^2 F(s, t)$. Substitution into the differential equation now yields

$$-\alpha(2\pi s)^2 F(s, t) = \frac{\partial}{\partial t} F(s, t). \quad (1)$$

b) The Fourier transform of $\exp(-\pi x^2)$ is $\exp(-\pi s^2)$. The differentiation theorem of Fourier transform theory may now be invoked to determine $F(s, t = 0)$, as follows:

$$F(s, t = 0) = \mathcal{F}\{f(x, t = 0)\} = \mathcal{F}\{d \exp(-\pi x^2)/dx\} = i2\pi s \exp(-\pi s^2). \quad (2)$$

c) The solution to Eq.(1) is readily found to be

$$F(s, t) = F(s, t = 0) \exp(-4\pi^2 \alpha s^2 t) = i2\pi s \exp(-\pi s^2) \exp(-4\pi^2 \alpha s^2 t). \quad (3)$$

d) We thus have

$$f(x, t) = \mathcal{F}^{-1}\{F(s, t)\} = \mathcal{F}^{-1}\{i2\pi s \exp[-\pi(1 + 4\pi\alpha t)s^2]\}$$

Differentiation theorem $\rightarrow = \frac{d}{dx} \mathcal{F}^{-1}\{\exp[-\pi(\sqrt{1 + 4\pi\alpha t} s)^2]\}$

Scaling theorem $\rightarrow = \frac{1}{\sqrt{1 + 4\pi\alpha t}} \frac{d}{dx} \exp\left[-\pi \left(\frac{x}{\sqrt{1 + 4\pi\alpha t}}\right)^2\right]$

$$= -\frac{2\pi x}{(1 + 4\pi\alpha t)^{3/2}} \exp\left(-\frac{\pi x^2}{1 + 4\pi\alpha t}\right). \quad (4)$$
