Problem 9) a) Imagine extending the rod to the negative x-axis, then imposing the initial condition $-f(-x)$ on the region $-\infty < x < 0$. The entire rod will then have an initial temperature distribution that is an odd function of x. As such, its temperature will remain zero at $x=0$ at all times, because the Fourier Transform of the initial condition contains only sine functions (no constant term and no cosine terms), which will remain zero at $x=0$ at all times $t \geq 0$. The solution to the problem is then obtained by a convolution between the initial condition and the impulse response, which was derived as $\frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{x^2}{4Dt})$ for an infinite rod having an arbitrary initial temperature distribution. We'll have:

$$T(x,t) = T(x,0) * \frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{x^2}{4Dt}) = \int_{-\infty}^{\infty} T(x',0) \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x-x')^2}{4Dt}\right] dx'$$

$$= -\frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{0} f(-x') \exp\left[-\frac{(x-x')^2}{4Dt}\right] dx' + \frac{1}{\sqrt{4\pi Dt}} \int_{0}^{\infty} f(x') \exp\left[-\frac{(x-x')^2}{4Dt}\right] dx'$$

$$\Rightarrow T(x,t) = \frac{1}{\sqrt{4\pi Dt}} \left\{ \int_0^{\infty} f(x') \exp\left[-\frac{(x-x')^2}{4Dt}\right] dx' - \int_0^{\infty} f(x') \exp\left[-\frac{(x+x')^2}{4Dt}\right] dx' \right\}$$

Change of variable from x' to $-x'$

Setting $f(x) = T_0$ in the above equation, we'll find:

$$T(x,t) = \frac{T_0}{\sqrt{4\pi Dt}} \left\{ \int_{-x}^{\infty} \exp(-\frac{y^2}{4Dt}) dy - \int_{x}^{\infty} \exp(-\frac{y^2}{4Dt}) dy \right\} = \frac{T_0}{\sqrt{4\pi Dt}} \int_{-x}^{x} \exp(-\frac{y^2}{4Dt}) dy$$

Change of variable: $y = x' - x$ Change of variable $y = x' + x$

$$= \frac{2T_0}{\sqrt{4\pi Dt}} \int_0^x \exp(-\frac{y^2}{4Dt}) dy = \frac{2T_0}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4Dt}}} \exp(-u^2) du = T_0 \, \mathrm{erf}\left(\frac{x}{\sqrt{4Dt}}\right).$$

b) In this case, we imagine extending the rod to the negative x-axis and imposing the initial condition $T(x,0) = f(-x)$ on the region $-\infty < x < 0$. Since the initial condition is now an even function of x, its Fourier representation will consist of a constant term plus cosine functions (but no sine functions). Therefore $\partial T(x,t)/\partial x \big|_{x=0}$ will be zero at all times $t \geq 0$. The solution is then found by convolving the initial condition with the impulse response:

$$T(x,t) = T(x,0) * \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right) = \frac{1}{\sqrt{4\pi D t}} \left\{ \int_{-\infty}^{0} f(-x') \exp\left[-\frac{(x-x')^2}{4Dt}\right] dx' \right.$$

$$\left. + \int_{0}^{\infty} f(x') \exp\left[-\frac{(x-x')^2}{4Dt}\right] dx' \right\} = \frac{1}{\sqrt{4\pi D t}} \int_{0}^{\infty} f(x') \left\{ \exp\left[-\frac{(x-x')^2}{4Dt}\right] + \exp\left[-\frac{(x+x')^2}{4Dt}\right] \right\} dx'$$

In this case, starting from a uniform initial temperature, $T(x,0) = T_0$, will yield:

$$T(x,t) = \frac{T_0}{\sqrt{4\pi D t}} \left\{ \int_{-x}^{\infty} e^{-\frac{y^2}{4Dt}} dy + \int_{x}^{\infty} e^{-\frac{y^2}{4Dt}} dy \right\} = \frac{T_0}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} e^{-y^2/4Dt} dy$$

$$= \frac{T_0}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = T_0 .$$

This is expected, of course, considering that the semi-infinite rod starts at $t=0$ with a uniform temperature, and does <u>not</u> lose any heat from its free end at $x=0$.