Problem 8) Writing the separable solution as $\psi(\boldsymbol{r},t) = f(r)g(\varphi)h(z)p(t)$, upon substitution into the wave equation and division by ψ, we find

$$v^2\left[\frac{f''(r)}{f(r)} + \frac{f'(r)}{rf(r)} + \frac{g''(\varphi)}{r^2 g(\varphi)} + \frac{h''(z)}{h(z)}\right] = \frac{p''(t) + \gamma p'(t)}{p(t)}. \tag{1}$$

Both sides of the above equation must now be equated to a negative constant $-c^2$, because otherwise one of the solutions for $p(t)$ will grow indefinitely with time, which is physically inadmissible. The left-hand side of Eq.(1) can be a constant only if its various terms that depend on r, φ, and z are separately equal to constants. We thus have

$$\frac{g''(\varphi)}{g(\varphi)} = -m^2 \quad \rightarrow \quad g(\varphi) = A_1\cos(m\varphi) + A_2\sin(m\varphi) \quad \rightarrow \quad g(\varphi) = A\cos(m\varphi + \varphi_0). \tag{2}$$

$$\frac{h''(z)}{h(z)} = -k_z^2 \quad \rightarrow \quad h(z) = B_1\sin(k_z z) + B_2\cos(k_z z) \quad \rightarrow \quad h(z) = B\sin(\ell\pi z/L). \tag{3}$$

$$\frac{f''(r)}{f(r)} + \frac{f'(r)}{rf(r)} - \frac{m^2}{r^2} - k_z^2 = -(c/v)^2 \quad \rightarrow \quad r^2 f''(r) + rf'(r) + (k_r^2 r^2 - m^2)f(r) = 0. \tag{4}$$

In the above equations, we have introduced the integers m and ℓ as the mode indices in the azimuthal and vertical directions φ and z, respectively. We have also defined in Eq.(4) the new parameter $k_r = \sqrt{(c/v)^2 - k_z^2} = \sqrt{(c/v)^2 - (\ell\pi/L)^2}$. The solutions to the Bessel equation are

$$f(r) = C_1 J_m(k_r r) + C_2 Y_m(k_r r). \tag{5}$$

Since the volume of interest contains the z-axis, for which $r = 0$, the term containing a Bessel function of the second kind $Y_m(\cdot)$ must vanish, that is $C_2 = 0$. Given that $\psi(\boldsymbol{r},t) = 0$ at $r = R$, we must have $k_r R = \rho_{mn}$, where ρ_{mn} is the n^{th} zero of $J_m(\rho)$. Consequently,

$$\left(\frac{c}{v}\right)^2 - \left(\frac{\ell\pi}{L}\right)^2 = \left(\frac{\rho_{mn}}{R}\right)^2 \qquad \rightarrow \qquad c^2 = \left[\left(\frac{v\rho_{mn}}{R}\right)^2 + \left(\frac{v\ell\pi}{L}\right)^2\right]. \tag{6}$$

The time-dependent factor $p(t)$ is thus seen to be the solution of the following equation:

$$p''(t) + \gamma p'(t) + c^2 p(t) = 0. \tag{7}$$

The solutions of Eq.(7) are in the form of $\exp(\eta t)$, where $\eta^2 + \gamma\eta + c^2 = 0$. consequently,

$$\eta_\pm = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - c^2}. \tag{8}$$

Had we chosen the initial separation constant to be positive (i.e., c^2 rather than $-c^2$), Eq.(8) would have yielded a positive value for η_+, which would have been physically untenable. With a negative separation constant, the two values of η will be either real and negative (over-damped), real, negative and equal (critically-damped), or complex conjugates with a negative real part (under-damped). The general solution of Eq.(7) thus acquires one of the following forms:

$$p(t) = \begin{cases} D_1\exp(\eta_+ t) + D_2\exp(\eta_- t)\,; & \text{(overdamped)} \\ D_1\exp(-½\gamma t) + D_2 t\exp(-½\gamma t)\,; & \text{(critically} - \text{damped)} \\ D\exp(-½\gamma t)\cos\left[\sqrt{(v\rho_{mn}/R)^2 + (v\ell\pi/L)^2 - (\gamma/2)^2}\;t + \chi_0\right]; & \text{(unerdamped).} \end{cases} \tag{9}$$

Thus the general solution in the underdamped case, for instance, is written as follows:

$$\psi(\boldsymbol{r},t) = \sum_m \sum_n \sum_\ell C_{mn\ell}\, J_m\left(\frac{\rho_{mn} r}{R}\right) \cos(m\varphi + \varphi_{mn\ell}) \sin(\ell\pi z/L) \\ \times \exp(-½\gamma t)\ \cos(\omega_{mn\ell} t + \chi_{mn\ell}). \qquad (10)$$

The unknown parameters $C_{mn\ell}$, $\varphi_{mn\ell}$, and $\chi_{mn\ell}$ must be determined from the initial conditions at $t = 0$.