Problem 7) Heat diffusion equation $D\dfrac{\partial^2 T(x,t)}{\partial x^2} = \dfrac{\partial T(x,t)}{\partial t}$

Separation of Variables: $T(x,t) = f(x)g(t) \Rightarrow D f''(x)g(t) = f(x)g'(t) \Rightarrow$

$$D\frac{f''(x)}{f(x)} = \frac{g'(t)}{g(t)} = \alpha \Rightarrow g'(t) = \alpha g(t) \Rightarrow g(t) = e^{\alpha t} \Rightarrow \alpha \text{ a negative constant.}$$

$$D\frac{f''(x)}{f(x)} = \alpha \Rightarrow f''(x) - \frac{\alpha}{D}f(x) = 0 \Rightarrow f(x) = A\sin\left(\sqrt{\tfrac{|\alpha|}{D}}\,x\right) + B\cos\left(\sqrt{\tfrac{|\alpha|}{D}}\,x\right).$$

a) $T(x=0,t) = T(x=L,t) = 0 \Rightarrow f(x) = A\sin\left(\frac{n\pi x}{L}\right) \Rightarrow \alpha = -\frac{n^2\pi^2 D}{L^2}$.

Therefore, $T(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 D}{L^2}t\right)$.

$$T(x,t=0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right); \quad 0 \leq x \leq L$$

Since the right-hand side is an odd function of x, we must make $T(x,0)$ also odd. $T(x,0)$ is defined only over $0 \leq x \leq L$; however, it can be extended arbitrarily into other regions of the x-axis; in particular, let's call $T(x,t=0) = h(x)$ and define $T(x,t=0) = -h(-x)$ when $-L \leq x < 0$.

To find the coefficients A_n, we multiply both sides of the preceding equation by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate over the interval $-L \leq x \leq +L$. We'll have:

$$-\int_{-L}^{0} h(-x)\sin\left(\frac{m\pi x}{L}\right)dx + \int_0^L h(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \int_{-L}^{+L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\Rightarrow 2\int_0^L h(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}\frac{1}{2}A_n\int_{-L}^{+L}\left\{\cos\left[\frac{(n-m)\pi x}{L}\right] - \cos\left[\frac{(n+m)\pi x}{L}\right]\right\}dx$$

The only time when the right-hand side is non-zero is when $n=m$, in which case

$\int_{-L}^{L} \cos\left[\frac{(n-m)\pi x}{L}\right] dx = 2L$, and we'll have

$$2\int_0^L h(x)\sin\left(\frac{m\pi x}{L}\right)dx = L A_m \Rightarrow A_m = \frac{2}{L}\int_0^L T(x,t=0)\sin\left(\frac{m\pi x}{L}\right)dx$$

b) $\frac{\partial}{\partial x}T(x=0,t) = \frac{\partial}{\partial x}T(x=L,t) = 0 \Rightarrow f(x) = B\cos\left(\frac{n\pi x}{L}\right) \Rightarrow \alpha = -\frac{n^2\pi^2 D}{L^2}.$

Therefore, $T(x,t) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 D}{L^2}t\right) \Rightarrow$

$$T(x,t) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 D}{L^2}t\right).$$

$$T(x,t=0) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right); \quad 0 \le x \le L.$$

Since the right-hand-side is an even function of x, we make the left-hand side even by defining $T(x,0) = h(x)$ and $T(x,0) = h(-x)$; $-L \le x < 0$.

Then, $\int_{-L}^{0} h(-x)\,dx + \int_0^L h(x)\,dx = \int_{-L}^{L} B_0\,dx + \sum_{n=1}^{\infty} B_n \cancelto{0}{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx} \Rightarrow$

$B_0 = \frac{1}{L}\int_0^L h(x)\,dx.$ That is, B_0 is the average temperature along the wire.

$$\int_{-L}^{0} h(-x)\cos\left(\frac{m\pi x}{L}\right)dx + \int_0^L h(x)\cos\left(\frac{m\pi x}{L}\right)dx = \cancelto{0}{\int_{-L}^{L} B_0\cos\left(\frac{m\pi x}{L}\right)dx} +$$

$$\sum_{n=1}^{\infty} B_n \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx \Rightarrow 2\int_0^L h(x)\cos\left(\frac{m\pi x}{L}\right)dx =$$

$$\sum_{n=1}^{\infty} \frac{1}{2}B_n \int_{-L}^{L}\left\{\cos\left[\frac{(n-m)\pi x}{L}\right] + \cos\left[\frac{(n+m)\pi x}{L}\right]\right\}dx = L B_m \Rightarrow B_m = \frac{2}{L}\int_0^L T(x,0)\cos\left(\frac{m\pi x}{L}\right)dx.$$