Problem 3) For an infinitely long, thin wire with no constraints at the far-away boundaries (located at $x = \pm\infty$), the eigen-solutions of the wave equation are $\exp[\mathrm{i}(kx \pm \omega t)]$. Substitution into the (one-dimensional) wave equation shows that $\omega = kv$. The general solution is written as the sum of two superposition integrals, as follows:

$$z(x,t) = \int_{-\infty}^{\infty} Z^{(+)}(k)\exp[\mathrm{i}(kx+\omega t)]\,\mathrm{d}k + \int_{-\infty}^{\infty} Z^{(-)}(k)\exp[\mathrm{i}(kx-\omega t)]\,\mathrm{d}k. \quad (1)$$

Defining the Fourier transforms of the initial conditions as $F(k)$ and $G(k)$, we will have

$$f(x) = \int_{-\infty}^{\infty} F(k)\exp(\mathrm{i}kx)\,\mathrm{d}k, \quad \text{where} \quad F(k) = (2\pi)^{-1}\int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}kx)\,\mathrm{d}x. \quad (2)$$

$$g(x) = \int_{-\infty}^{\infty} G(k)\exp(\mathrm{i}kx)\,\mathrm{d}k, \quad \text{where} \quad G(k) = (2\pi)^{-1}\int_{-\infty}^{\infty} g(x)\exp(-\mathrm{i}kx)\,\mathrm{d}x. \quad (3)$$

A comparison with Eq.(1) at $t = 0$ reveals that the following identities must be satisfied:

$$F(k) = Z^{(+)}(k) + Z^{(-)}(k). \quad (4)$$

$$G(k) = \mathrm{i}\omega Z^{(+)}(k) - \mathrm{i}\omega Z^{(-)}(k). \quad (5)$$

Solving the above equations for $Z^{(+)}$ and $Z^{(-)}$, we find

$$Z^{(\pm)}(k) = \tfrac{1}{2}F(k) \pm \tfrac{1}{2\mathrm{i}\omega}G(k). \quad (6)$$

Substitution into Eq.(1) and the invocation of the identity $\omega = kv$ now yields

$$z(x,t) = \tfrac{1}{2}f(x+vt) + \tfrac{1}{2}f(x-vt) + \int_{-\infty}^{\infty} G(k)[\sin(kvt)/(kv)]\exp(\mathrm{i}kx)\,\mathrm{d}k. \quad (7)$$

The last term in the above equation is the inverse Fourier transform of the product of $G(k)$ and $\sin(kvt)/(kv)$. The inverse transforms of these functions are $g(x)$ and $(\pi/v)\mathrm{rect}(x/2vt)$, respectively. The inverse Fourier integral in Eq.(7) may thus be written as the convolution between the individual inverse transforms, namely,

$$(2\pi)^{-1}g(x) * (\pi/v)\mathrm{rect}(x/2vt) = \frac{1}{2v}\int_{-\infty}^{\infty} g(x')\mathrm{rect}\left(\frac{x-x'}{2vt}\right)\mathrm{d}x' = (1/2v)\int_{x-vt}^{x+vt} g(x')\mathrm{d}x'. \quad (8)$$

Substituting the above result into Eq.(7), we finally arrive at the desired D'Alembert formula,

$$z(x,t) = \tfrac{1}{2}f(x+vt) + \tfrac{1}{2}f(x-vt) + (1/2v)\int_{x-vt}^{x+vt} g(x')\mathrm{d}x'. \quad (9)$$

Digression: The following formulas have been used in the above derivations:

1) $$\mathcal{F}\{\mathrm{rect}(x/2vt)\} = (2\pi)^{-1}\int_{-vt}^{vt}\exp(-\mathrm{i}kx)\,\mathrm{d}x = \frac{\exp(-\mathrm{i}kvt)-\exp(\mathrm{i}kvt)}{-\mathrm{i}2\pi k} = (v/\pi)\sin(kvt)/(kv). \quad (10)$$

2) $$\begin{aligned}\int_{-\infty}^{\infty} G(k)H(k)\exp(\mathrm{i}kx)\,\mathrm{d}k &= \int_{-\infty}^{\infty}\left\{(2\pi)^{-1}\int_{-\infty}^{\infty} g(x')\exp(-\mathrm{i}kx')\,\mathrm{d}x'\right\}H(k)\exp(\mathrm{i}kx)\,\mathrm{d}k \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} g(x')\left\{\int_{-\infty}^{\infty} H(k)\exp[\mathrm{i}k(x-x')]\,\mathrm{d}k\right\}\mathrm{d}x' \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} g(x')h(x-x')\mathrm{d}x' = (2\pi)^{-1}g(x) * h(x). \quad (11)\end{aligned}$$
