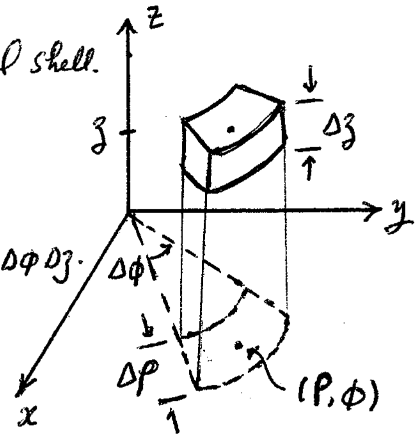
**Problem 2)** a) Small Volume Cut from a Cylindrical shell.




Front and back facets:

$$\frac{\partial\psi(\rho+\frac{\Delta\rho}{2},\phi,z)}{\partial\rho}(\rho+\frac{\Delta\rho}{2})\Delta\phi\Delta z-\frac{\partial\psi(\rho-\frac{\Delta\rho}{2},\phi,z)}{\partial\rho}(\rho-\frac{\Delta\rho}{2})\Delta\phi\Delta z.$$

Left and right facets:

$$\frac{\partial\psi(\rho,\phi+\frac{\Delta\phi}{2},z)}{\rho\partial\phi}\Delta\rho\Delta z-\frac{\partial\psi(\rho,\phi-\frac{\Delta\phi}{2},z)}{\rho\partial\phi}\Delta\rho\Delta z.$$

Top and bottom facets:

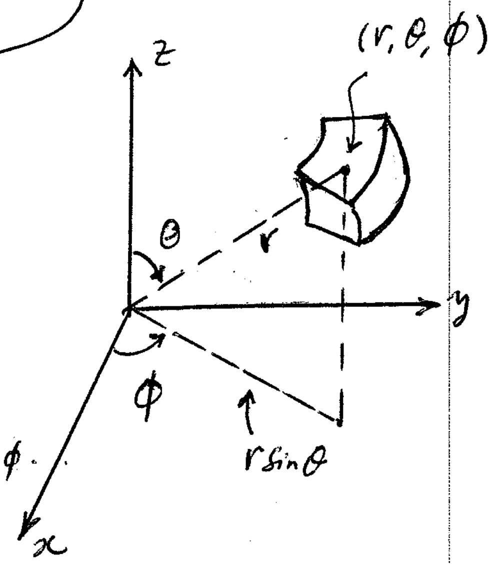
$$\frac{\partial\psi(\rho,\phi,z+\frac{\Delta z}{2})}{\partial z}\rho\Delta\rho\Delta\phi-\frac{\partial\psi(\rho,\phi,z-\frac{\Delta z}{2})}{\partial z}\rho\Delta\rho\Delta\phi.$$

Adding up the above contributions to the net outflow, then normalizing by the volume $\Delta V=\rho\Delta\rho\Delta\phi\Delta z$ of the element, we will have:

$$\nabla^2\psi(\rho,\phi,z)=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}\Longrightarrow$$

$$\nabla^2\psi(\rho,\phi,z)=\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}$$ ✓

b) Small Volume Cut from a spherical shell.




Front and back facets:

$$\frac{\partial\psi(r+\frac{\Delta r}{2},\theta,\phi)}{\partial r}(r+\frac{\Delta r}{2})^2\sin\theta\Delta\theta\Delta\phi$$

$$-\frac{\partial\psi(r-\frac{\Delta r}{2},\theta,\phi)}{\partial r}(r-\frac{\Delta r}{2})^2\sin\theta\Delta\theta\Delta\phi.$$

Left and right facets:

$$\frac{\partial\psi(r,\theta,\phi+\frac{\Delta\phi}{2})}{r\sin\theta\,\partial\phi}\,r\Delta r\Delta\theta-\frac{\partial\psi(r,\theta,\phi-\frac{\Delta\phi}{2})}{r\sin\theta\,\partial\phi}\,r\Delta r\Delta\theta$$

Top and bottom facets:

$$\frac{\partial\psi(r,\theta+\frac{\Delta\theta}{2},\phi)}{r\,\partial\theta}\,r\sin(\theta+\tfrac{\Delta\theta}{2})\Delta\phi\Delta r-\frac{\partial\psi(r,\theta-\frac{\Delta\theta}{2},\phi)}{r\,\partial\theta}\,r\sin(\theta-\tfrac{\Delta\theta}{2})\Delta\phi\Delta r$$

Adding up the above contributions to the net outflow, then normalizing by the volume $\Delta V = r^2\sin\theta\,\Delta\theta\Delta r\Delta\phi$ of the element, we'll find:

$$\nabla^2\psi(r,\theta,\phi)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}.$$

Note that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)=\frac{1}{r^2}\left(2r\frac{\partial\psi}{\partial r}+r^2\frac{\partial^2\psi}{\partial r^2}\right)=\frac{1}{r}\left(2\frac{\partial\psi}{\partial r}+r\frac{\partial^2\psi}{\partial r^2}\right)$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left(\psi+r\frac{\partial\psi}{\partial r}\right)=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).$$