

**Problem 22)** Let  $G(s) = \int_{-\infty}^{\infty} g(x) \exp(-i2\pi sx) dx$  be the Fourier transform of  $g(x)$ . The inverse Fourier transform relation, namely,  $g(x) = \int_{-\infty}^{\infty} G(s) \exp(i2\pi sx) ds$ , when differentiated with respect to  $x$ , yields  $g'(x) = \int_{-\infty}^{\infty} (i2\pi s)G(s) \exp(i2\pi sx) ds$ , which indicates that the Fourier transform of  $g'(x)$  is  $(i2\pi s)G(s)$ . Also, the Fourier transform of the right-hand side of the differential equation can be obtained by direct integration, as follows:

$$\begin{aligned} \mathcal{F}\{\text{rect}(x) \cos(2\pi s_0 x)\} &= \int_{-1/2}^{1/2} \cos(2\pi s_0 x) \exp(-i2\pi sx) dx \\ &= \frac{1}{2} \int_{-1/2}^{1/2} \exp[-i2\pi(s - s_0)x] dx + \frac{1}{2} \int_{-1/2}^{1/2} \exp[-i2\pi(s + s_0)x] dx \\ &= \frac{\exp[-i\pi(s-s_0)] - \exp[i\pi(s-s_0)]}{-i4\pi(s-s_0)} + \frac{\exp[-i\pi(s+s_0)] - \exp[i\pi(s+s_0)]}{-i4\pi(s+s_0)} \\ &= \frac{\sin[\pi(s-s_0)]}{2\pi(s-s_0)} + \frac{\sin[\pi(s+s_0)]}{2\pi(s+s_0)}. \end{aligned} \quad (1)$$

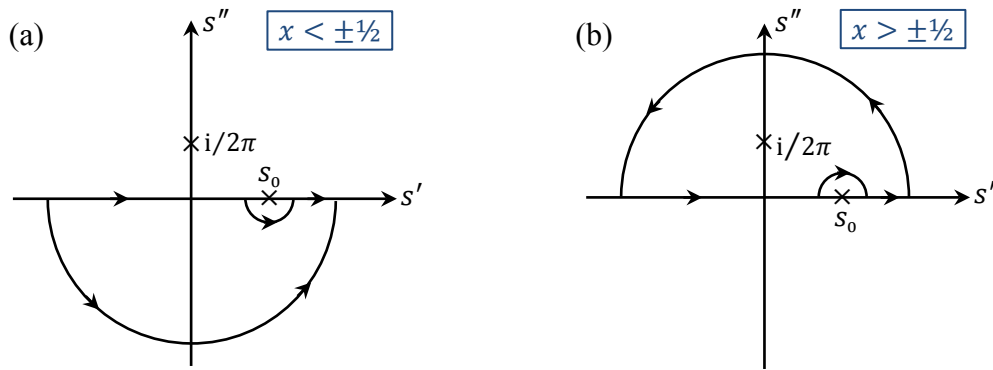
The Fourier transform of the differential equation may thus be written as follows:

$$\begin{aligned} (i2\pi s)G(s) + G(s) &= \frac{\sin[\pi(s-s_0)]}{2\pi(s-s_0)} + \frac{\sin[\pi(s+s_0)]}{2\pi(s+s_0)} \\ \rightarrow G(s) &= \frac{\sin[\pi(s-s_0)]}{2\pi(s-s_0)(1+i2\pi s)} + \frac{\sin[\pi(s+s_0)]}{2\pi(s+s_0)(1+i2\pi s)}. \end{aligned} \quad (2)$$

The last step is to find the inverse Fourier transform of  $G(s)$ . Considering that each sine function is a linear combination of two complex exponentials, we write

$$\begin{aligned} g(x) = \mathcal{F}^{-1}\{G(s)\} &= \int_{-\infty}^{\infty} \frac{\exp[i\pi(s-s_0)+i2\pi sx]}{i4\pi(s-s_0)(1+i2\pi s)} ds - \int_{-\infty}^{\infty} \frac{\exp[-i\pi(s-s_0)+i2\pi sx]}{i4\pi(s-s_0)(1+i2\pi s)} ds \\ &+ \int_{-\infty}^{\infty} \frac{\exp[i\pi(s+s_0)+i2\pi sx]}{i4\pi(s+s_0)(1+i2\pi s)} ds - \int_{-\infty}^{\infty} \frac{\exp[-i\pi(s+s_0)+i2\pi sx]}{i4\pi(s+s_0)(1+i2\pi s)} ds. \end{aligned} \quad (3)$$

The four integrals in Eq.(3) should be evaluated in the complex  $s$ -plane along the contours shown in the figure below. The exponential factors appearing in the integrands are in the form of  $\exp[i2\pi(x \pm \frac{1}{2})s \pm i\pi s_0]$ . Depending on  $x$  being greater than or less than  $\pm \frac{1}{2}$ , the contours must be closed in the upper-half or lower-half of the  $s$ -plane. Each integrand has a simple pole at  $s = i/2\pi$ , and a second (also simple) pole at either  $s = s_0$  or  $s = -s_0$ .



The first integral in Eq.(3) should be evaluated in the lower half of the complex  $s$ -plane when  $x < -\frac{1}{2}$ , and in the upper half when  $x > -\frac{1}{2}$ , as shown in the figure. The poles of the integrand are located at  $s = s_0$  and  $s = i/2\pi$ . Therefore,

$$\int_{-\infty}^{\infty} \frac{\exp[i\pi(s-s_0)+i2\pi sx]}{-8\pi^2(s-s_0)[s-(i/2\pi)]} ds = \begin{cases} \frac{-i\pi \exp(i2\pi s_0 x)}{-8\pi^2[s_0-(i/2\pi)]}; & x < -\frac{1}{2}, \\ \frac{i\pi \exp(i2\pi s_0 x)}{-8\pi^2[s_0-(i/2\pi)]} + \frac{i2\pi \exp\{i\pi[(i/2\pi)-s_0]-x\}}{-8\pi^2[(i/2\pi)-s_0]}; & x > -\frac{1}{2}. \end{cases} \quad (4)$$

The second integral is evaluated in the lower half-plane when  $x < \frac{1}{2}$ , and in the upper half-plane when  $x > \frac{1}{2}$ . The poles of the integrand are at  $s = s_0$  and  $s = i/2\pi$ . Therefore,

$$\int_{-\infty}^{\infty} \frac{\exp[-i\pi(s-s_0)+i2\pi sx]}{-8\pi^2(s-s_0)[s-(i/2\pi)]} ds = \begin{cases} \frac{-i\pi \exp(i2\pi s_0 x)}{-8\pi^2[s_0-(i/2\pi)]}; & x < \frac{1}{2}, \\ \frac{i\pi \exp(i2\pi s_0 x)}{-8\pi^2[s_0-(i/2\pi)]} + \frac{i2\pi \exp\{-i\pi[(i/2\pi)-s_0]-x\}}{-8\pi^2[(i/2\pi)-s_0]}; & x > \frac{1}{2}. \end{cases} \quad (5)$$

The third integral is evaluated in the lower half-plane when  $x < -\frac{1}{2}$ , and in the upper half-plane when  $x > -\frac{1}{2}$ . The poles of the integrand are at  $s = -s_0$  and  $s = i/2\pi$ . Therefore,

$$\int_{-\infty}^{\infty} \frac{\exp[i\pi(s+s_0)+i2\pi sx]}{-8\pi^2(s+s_0)[s-(i/2\pi)]} ds = \begin{cases} \frac{-i\pi \exp(-i2\pi s_0 x)}{8\pi^2[s_0+(i/2\pi)]}; & x < -\frac{1}{2}, \\ \frac{i\pi \exp(-i2\pi s_0 x)}{8\pi^2[s_0+(i/2\pi)]} + \frac{i2\pi \exp\{i\pi[(i/2\pi)+s_0]-x\}}{-8\pi^2[(i/2\pi)+s_0]}; & x > -\frac{1}{2}. \end{cases} \quad (6)$$

The fourth integral is evaluated in the lower half-plane when  $x < \frac{1}{2}$ , and in the upper half-plane when  $x > \frac{1}{2}$ . The poles of the integrand are at  $s = -s_0$  and  $s = i/2\pi$ . Therefore,

$$\int_{-\infty}^{\infty} \frac{\exp[-i\pi(s+s_0)+i2\pi sx]}{-8\pi^2(s+s_0)[s-(i/2\pi)]} ds = \begin{cases} \frac{-i\pi \exp(-i2\pi s_0 x)}{8\pi^2[s_0+(i/2\pi)]}; & x < \frac{1}{2}, \\ \frac{i\pi \exp(-i2\pi s_0 x)}{8\pi^2[s_0+(i/2\pi)]} + \frac{i2\pi \exp\{-i\pi[(i/2\pi)+s_0]-x\}}{-8\pi^2[(i/2\pi)+s_0]}; & x > \frac{1}{2}. \end{cases} \quad (7)$$

Returning to Eq.(3), we now combine the integrals given in Eqs.(4)-(7) to determine  $g(x)$  over the entire  $x$ -axis (from  $-\infty$  to  $\infty$ ), as follows:

$$g(x) = \begin{cases} 0; & x < -\frac{1}{2}, \\ \frac{i\pi \exp(-i2\pi s_0 x)}{4\pi^2[s_0+(i/2\pi)]} - \frac{i\pi \exp(i2\pi s_0 x)}{4\pi^2[s_0-(i/2\pi)]} + \frac{\exp(-x-\frac{1}{2}-i\pi s_0)}{-2(1+i2\pi s_0)} + \frac{\exp(-x-\frac{1}{2}+i\pi s_0)}{-2(1-i2\pi s_0)}; & |x| < \frac{1}{2}, \\ \frac{\exp(-x-\frac{1}{2}-i\pi s_0) - \exp(-x+\frac{1}{2}+i\pi s_0)}{-2(1+i2\pi s_0)} + \frac{\exp(-x-\frac{1}{2}+i\pi s_0) - \exp(-x+\frac{1}{2}-i\pi s_0)}{-2(1-i2\pi s_0)}; & x > \frac{1}{2}. \end{cases} \quad (8)$$

Further simplifications yield

$$g(x) = \begin{cases} 0; & x < -\frac{1}{2}, \\ \frac{i\pi\{[s_0-(i/2\pi)] \exp(-i2\pi s_0 x) - [s_0+(i/2\pi)] \exp(i2\pi s_0 x)\}}{1+4\pi^2 s_0^2} - \frac{\exp(-x-\frac{1}{2})[(1-i2\pi s_0) \exp(-i\pi s_0) + (1+i2\pi s_0) \exp(i\pi s_0)]}{2(1+4\pi^2 s_0^2)}; & |x| < \frac{1}{2} \\ -\frac{(1-i2\pi s_0)[\exp(-x-\frac{1}{2}-i\pi s_0) - \exp(-x+\frac{1}{2}+i\pi s_0)] + (1+i2\pi s_0)[\exp(-x-\frac{1}{2}+i\pi s_0) - \exp(-x+\frac{1}{2}-i\pi s_0)]}{2(1+4\pi^2 s_0^2)}; & x > \frac{1}{2}. \end{cases}$$

$$g(x) = \begin{cases} 0; & x < -\frac{1}{2}, \\ \frac{\cos(2\pi s_0 x) + 2\pi s_0 \sin(2\pi s_0 x) - \exp(-x-\frac{1}{2})[\cos(\pi s_0) - 2\pi s_0 \sin(\pi s_0)]}{1+4\pi^2 s_0^2}; & |x| < \frac{1}{2}, \\ \frac{-\exp(-x-\frac{1}{2})[(1-i2\pi s_0) \exp(-i\pi s_0) + (1+i2\pi s_0) \exp(i\pi s_0)] + \exp(-x+\frac{1}{2})[(1-i2\pi s_0) \exp(i\pi s_0) + (1+i2\pi s_0) \exp(-i\pi s_0)]}{2(1+4\pi^2 s_0^2)}; & x > \frac{1}{2}. \end{cases}$$

$$g(x) = \begin{cases} 0; & x < -\frac{1}{2}, \\ \frac{\cos(2\pi s_0 x) + 2\pi s_0 \sin(2\pi s_0 x) - [\cos(\pi s_0) - 2\pi s_0 \sin(\pi s_0)] \exp(-x-\frac{1}{2})}{1+4\pi^2 s_0^2}; & |x| < \frac{1}{2}, \\ \frac{[\cos(\pi s_0) + 2\pi s_0 \sin(\pi s_0)] \exp(-x+\frac{1}{2}) - [\cos(\pi s_0) - 2\pi s_0 \sin(\pi s_0)] \exp(-x-\frac{1}{2})}{1+4\pi^2 s_0^2}; & x > \frac{1}{2}. \end{cases} \quad (9)$$

As expected, the above solution is continuous at both extremes of the excitation function, namely, at  $x = \pm\frac{1}{2}$ .

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