Problem 21) The stationary-phase approximation applies to $I = \int_a^b f(x) e^{i\eta g(x)} dx$. At each stationary-point of $g(x)$, i.e., a point such as $x_0$ where $g'(x_0)=0$, the contribution of that stationary point to the integral is given by $\sqrt{\frac{2\pi}{\eta |g''(x_0)|}}\, f(x_0)\, e^{i\eta g(x_0)}\, e^{\pm i\pi/4}$, with $\pm$ sign depending on whether $g''(x_0)$ is $>0$ or $<0$.

In the present problem $J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta} d\theta$. Thus $x$ plays the role of $\eta$, which becomes large when the asymptotic behavior of $J_0(x)$ is desired. Since $g(\theta)=\sin\theta$, we'll have $g'(\theta)=\cos\theta=0$

$\Rightarrow$ $\theta_0 = \pi/2$ and $\theta_1 = 3\pi/2$, so there are Two stationary points. We must then calculate the contribution of each stationary point to the integral as follows:

$\theta_0 = \pi/2$ ; $g(\theta_0) = \sin(\pi/2) = 1$ ; $g''(\theta_0) = -\sin(\pi/2) = -1$ $\Rightarrow$

Contribution of $\theta_0$ to the integral: $\sqrt{\frac{2\pi}{x|-1|}}\, e^{ix}\, e^{-i\pi/4} = \sqrt{\frac{2\pi}{x}}\, e^{i(x-\pi/4)}$.

$\theta_0 = \frac{3\pi}{2}$ ; $g(\theta_0) = \sin(3\pi/2) = -1$; $g''(\theta_0) = -\sin(\frac{3\pi}{2}) = +1$ $\Rightarrow$

Contribution of $\theta_1$ to the integral: $\sqrt{\frac{2\pi}{x|+1|}}\, e^{-ix}\, e^{+i\pi/4} = \sqrt{\frac{2\pi}{x}}\, e^{-i(x-\pi/4)}$.

Consequently, $J_0(x) \sim \frac{1}{2\pi}\left\{\sqrt{\frac{2\pi}{x}}\, e^{i(x-\pi/4)} + \sqrt{\frac{2\pi}{x}}\, e^{-i(x-\pi/4)}\right\} = \sqrt{\frac{2}{\pi x}}\cos(x-\pi/4)$.