Problem 20)

$$f(x) = x^s \sum_{k=0}^{\infty} A_k x^k$$

$$f'(x) = \sum_{k=0}^{\infty}(k+s)A_k x^{k+s-1}$$

$$f''(x) = \sum_{k=0}^{\infty}(k+s)(k+s-1)A_k x^{k+s-2}$$

Airy's equation: $f''(x) - xf(x) = \sum_{k=0}^{\infty}(k+s)(k+s-1)A_k x^{k+s-2} - \sum_{k=0}^{\infty} A_k x^{k+s+1} = 0.$

Defining $k' = k-3$, then switching the dummy of the summation back to k, we will have

$$\sum_{k'=-3}^{\infty}(k'+s+3)(k'+s+2)A_{k'+3}x^{k'+s+1} - \sum_{k=0}^{\infty} A_k x^{k+s+1}$$

$$= s(s-1)A_0 x^{s-2} + s(s+1)A_1 x^{s-1} + (s+1)(s+2)A_2 x^s$$

$$+ \sum_{k=0}^{\infty}[(k+s+3)(k+s+2)A_{k+3} - A_k]x^{k+s+1} = 0.$$

Indicial equations: $s(s-1)A_0 = 0$; $s(s+1)A_1 = 0$; $(s+1)(s+2)A_2 = 0.$

Solutions of the indicial equations:

i) $s = 1$, A_0 arbitrary, $A_1 = A_2 = 0.$

ii) $s = -2$, A_2 arbitrary, $A_0 = A_1 = 0.$

iii) $s = 0$, A_0 and A_1 arbitrary, $A_2 = 0.$

iv) $s = -1$, A_1 and A_2 arbitrary, $A_0 = 0.$

Recursion relation: $A_{k+3} = \frac{A_k}{(k+s+2)(k+s+3)}.$

First solution of Airy's equation ($\boldsymbol{s = 1}$): $A_{k+3} = \frac{A_k}{(k+3)(k+4)}$, $k = 0, 3, 6, 9, \cdots.$

$$A_3 = \frac{A_0}{3\cdot 4} = \frac{2}{4!}A_0; \qquad A_6 = \frac{A_3}{6\cdot 7} = \frac{A_0}{3\cdot 4\cdot 6\cdot 7} = \frac{2\cdot 5}{7!}A_0; \qquad A_9 = \frac{A_6}{9\cdot 10} = \frac{2\cdot 5\cdot 8}{10!}A_0; \qquad \cdots$$

Therefore, $A_{3n} = \frac{(3-1)\cdot(6-1)\cdot(9-1)\cdots(3n-1)}{(3n+1)!}A_0 = \frac{3^n(1-⅓)(2-⅓)(3-⅓)\cdots(n-⅓)}{(3n+1)!}A_0 = \frac{3^n(n-⅓)!}{(3n+1)!}A_0.$

$$f_1(x) = x\left[1 + \sum_{n=1}^{\infty}\frac{(n-⅓)!\,(3^{⅓}x)^{3n}}{(3n+1)!}\right].$$

Second solution of Airy's equation ($\boldsymbol{s = -2}$): $A_{k+3} = \frac{A_k}{k(k+1)}$, $k = 2, 5, 8, 11, \cdots.$

$$A_5 = \frac{A_2}{2\cdot 3} = \frac{1}{3!}A_2; \qquad A_8 = \frac{A_5}{5\cdot 6} = \frac{A_2}{2\cdot 3\cdot 5\cdot 6} = \frac{1\cdot 4}{6!}A_2; \qquad A_{11} = \frac{A_8}{8\cdot 9} = \frac{1\cdot 4\cdot 7}{9!}A_2; \qquad \cdots$$

Therefore, $A_{3n+2} = \frac{(3-2)\cdot(6-2)\cdot(9-2)\cdots(3n-2)}{(3n)!}A_2 = \frac{3^n(1-⅔)(2-⅔)(3-⅔)\cdots(n-⅔)}{(3n)!}A_2 = \frac{3^n(n-⅔)!}{(3n)!}A_2.$

$$f_2(x) = 1 + \sum_{n=1}^{\infty}\frac{(n-⅔)!\,(3^{⅓}x)^{3n}}{(3n)!}.$$

The remaining solutions of the indicial equations (associated with $s = 0$ and $s = -1$) do not yield any new solutions for the Airy equation. For example, in the case of $s = 0$, we will have

$$A_{k+3} = \frac{A_k}{(k+2)(k+3)}, \qquad k = 0, 3, 6, 9, \cdots \quad \text{and also} \quad k = 1, 4, 7, 10, \cdots$$

The first series ($k = 0, 3, 6, 9, \cdots$) yields $f_2(x)$, while the second ($k = 1, 4, 7, 10, \cdots$) yields $f_1(x)$, so that the general solution will be $f(x) = A_0 f_2(x) + A_1 f_1(x)$. Similarly, in the case of $s = -1$, we will have

$$A_{k+3} = \frac{A_k}{(k+1)(k+2)}, \qquad k = 1, 4, 7, 10, \cdots \quad \text{and also} \quad k = 2, 5, 8, 11, \cdots$$

The first series ($k = 1, 4, 7, 10, \cdots$) yields $f_2(x)$, while the second ($k = 2, 5, 8, 11, \cdots$) yields $f_1(x)$, so that the general solution will be $f(x) = A_1 f_2(x) + A_2 f_1(x)$.
