

**Problem 17)** To substitute  $J_n(\alpha x)$  for  $f(x)$  in  $x^2 f''(x) + x f'(x) + (\alpha^2 x^2 - n^2) f(x)$  we write

$$J_n(\alpha x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x / 2)^{2m+n}}{m!(n+m)!},$$

$$J'_n(\alpha x) = (\alpha / 2) \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(\alpha x / 2)^{2m+n-1}}{m!(n+m)!},$$

$$J''_n(\alpha x) = (\alpha / 2)^2 \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(2m+n-1)(\alpha x / 2)^{2m+n-2}}{m!(n+m)!}.$$

We will have

$$\begin{aligned} & x^2 J''_n(\alpha x) + x J'_n(\alpha x) + (\alpha^2 x^2 - n^2) J_n(\alpha x) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(2m+n-1)(\alpha x / 2)^{2m+n}}{m!(n+m)!} + \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(\alpha x / 2)^{2m+n}}{m!(n+m)!} \\ & \quad + (\alpha^2 x^2 - n^2) \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x / 2)^{2m+n}}{m!(n+m)!} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m [(2m+n)^2 - n^2](\alpha x / 2)^{2m+n}}{m!(n+m)!} + \alpha^2 x^2 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x / 2)^{2m+n}}{m!(n+m)!} \\ &= 4 \sum_{m=0}^{\infty} \frac{(-1)^m m(n+m)(\alpha x / 2)^{2m+n}}{m!(n+m)!} + 4 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x / 2)^{2m+n+2}}{m!(n+m)!} \\ &= 4 \sum_{m=1}^{\infty} \frac{(-1)^m (\alpha x / 2)^{2m+n}}{(m-1)!(n+m-1)!} + 4 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x / 2)^{2m+n+2}}{m!(n+m)!} \\ & \xrightarrow{k=m-1} 4 \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (\alpha x / 2)^{2(k+1)+n}}{k!(n+k)!} + 4 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x / 2)^{2m+n+2}}{m!(n+m)!} \\ &= -4 \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha x / 2)^{2k+n+2}}{k!(n+k)!} + 4 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x / 2)^{2m+n+2}}{m!(n+m)!} = 0. \end{aligned}$$

We conclude that  $J_n(\alpha x)$  satisfies the Bessel equation  $x^2 f''(x) + x f'(x) + (\alpha^2 x^2 - n^2) f(x) = 0$ .

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