Problem 15) $f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i[x\sin\theta-n\theta]}\,d\theta$

$$\frac{d^k f(x)}{dx^k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}(i\sin\theta)^k e^{i[x\sin\theta-n\theta]}\,d\theta$$

$$k^{th}\text{ Taylor Series Coefficient}=\frac{1}{k!}\left.\frac{d^k f(x)}{dx^k}\right|_{x=0}=\frac{1}{2\pi k!}\int_{-\pi}^{\pi}\left(\frac{e^{i\theta}-e^{-i\theta}}{2}\right)^k e^{-in\theta}\,d\theta$$

$$=\frac{1}{2\pi k!}\cdot\frac{1}{2^k}\int_{-\pi}^{\pi}\underbrace{\sum_{m=0}^{k}\binom{k}{m}e^{i(k-m)\theta}(-1)^m e^{-im\theta}}_{\text{Binomial expansion}}e^{-in\theta}\,d\theta$$

$$=\frac{1}{2\pi k!\,2^k}\sum_{m=0}^{k}(-1)^m\binom{k}{m}\int_{-\pi}^{\pi}e^{i(k-2m-n)\theta}\,d\theta$$

The integral in the above expression is zero unless $k-2m-n=0$, in which case it is equal to 2π. The sum over m thus reduces to a single term corresponding to $k=2m+n$. We will have:

$$k^{th}\text{ Taylor series Coeff.}=\frac{1}{\cancel{k!}\,2^k}(-1)^m\frac{\cancel{k!}}{m!\,(k-m)!}=\frac{(-1)^m}{2^k m!\,(m+n)!}.$$

The Taylor series expansion of $f(x)=J_n(x)$ is thus given by

$$J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m x^k}{2^k\,m!\,(n+m)!}=\sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{2m+n}}{m!\,(n+m)!}.$$

✓