Problem 14) $f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x\sin\theta-n\theta)}\,d\theta \Rightarrow$

$$f'(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} i\sin\theta\, e^{i(x\sin\theta-n\theta)}\,d\theta \Rightarrow f''(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(-\sin^2\theta)\, e^{i(x\sin\theta-n\theta)}\,d\theta$$

Bessel's equation: $x^2 f''(x)+x f'(x)+(x^2-n^2)f(x)=0 \Rightarrow$

$$\int_{-\pi}^{\pi}(x^2\cos^2\theta-n^2)\,e^{i(x\sin\theta-n\theta)}\,d\theta+\int_{-\pi}^{\pi}(ix\sin\theta)\,e^{i(x\sin\theta-n\theta)}\,d\theta=0$$

First integral: $\int_{-\pi}^{\pi}(x\cos\theta+n)(x\cos\theta-n)\,e^{i(x\sin\theta-n\theta)}\,d\theta$

integration by parts $\rightarrow$ $$=\cancelto{0}{-i(x\cos\theta+n)\,e^{i(x\sin\theta-n\theta)}\Big|_{-\pi}^{\pi}}-\int_{-\pi}^{\pi}(ix\sin\theta)\,e^{i(x\sin\theta-n\theta)}\,d\theta$$

This is clearly cancelled out by the 2nd integral, resulting in the satisfaction of Bessel's equation.

a) $$J_0'(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} i\sin\theta\, e^{ix\sin\theta}\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos\theta-\cos\theta+i\sin\theta)\,e^{ix\sin\theta}\,d\theta$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\theta\, e^{ix\sin\theta}\,d\theta-\frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos\theta-i\sin\theta)\,e^{ix\sin\theta}\,d\theta$$

$$=\cancelto{0}{\frac{1}{i2\pi x}e^{ix\sin\theta}\Big|_{-\pi}^{\pi}}-\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\theta}e^{ix\sin\theta}\,d\theta=-\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(x\sin\theta-\theta)}\,d\theta=-J_1(x).$$ ✓

b) $$J_{n-1}(x)-J_{n+1}(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i[x\sin\theta-(n-1)\theta]}\,d\theta-\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i[x\sin\theta-(n+1)\theta]}\,d\theta$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i[x\sin\theta-n\theta]}(e^{i\theta}-e^{-i\theta})\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}2i\sin\theta\, e^{i[x\sin\theta-n\theta]}\,d\theta$$

$$=2\frac{d}{dx}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i[x\sin\theta-n\theta]}\,d\theta\right\}=2J_n'(x).$$ ✓

c) $$x\left[J_{n-1}(x)+J_{n+1}(x)\right]=\frac{x}{2\pi}\int_{-\pi}^{\pi}e^{i[x\sin\theta-(n-1)\theta]}d\theta+\frac{x}{2\pi}\int_{-\pi}^{\pi}e^{i[x\sin\theta-(n+1)\theta]}d\theta$$

$$=\frac{x}{2\pi}\int_{-\pi}^{\pi}e^{i[x\sin\theta-n\theta]}\left(e^{i\theta}+e^{-i\theta}\right)d\theta=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos\theta\, e^{i[x\sin\theta-n\theta]}d\theta$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x\cos\theta-n+n)e^{i[x\sin\theta-n\theta]}d\theta=\underbrace{\cancel{\frac{1}{i\pi}e^{i[x\sin\theta-n\theta]}\Big|_{-\pi}^{\pi}}}_{0}+\frac{n}{\pi}\int_{-\pi}^{\pi}e^{i[x\sin\theta-n\pi]}d\theta$$

$$=2nJ_n(x).$$ ✓