Problem 13) $f(x)=\int_{-\pi}^{\pi} e^{ix\sin\theta}\,d\theta \Rightarrow f'(x)=\int_{-\pi}^{\pi} i\sin\theta\, e^{ix\sin\theta}\,d\theta$

$$\Rightarrow f''(x)=-\int_{-\pi}^{\pi}\sin^2\theta\,\exp(ix\sin\theta)\,d\theta.$$

Bessel's equation ($n=0$): $x^2 f''(x)+xf'(x)+x^2f(x)=0 \Rightarrow$

$$-\int_{-\pi}^{\pi}x^2\sin^2\theta\, e^{ix\sin\theta}\,d\theta+\int_{-\pi}^{\pi}ix\sin\theta\, e^{ix\sin\theta}\,d\theta+\int_{-\pi}^{\pi}x^2\, e^{ix\sin\theta}\,d\theta$$

$$=\int_{-\pi}^{\pi}\left[x^2(1-\sin^2\theta)+ix\sin\theta\right]e^{ix\sin\theta}\,d\theta=\int_{-\pi}^{\pi}x^2\cos^2\theta\, e^{ix\sin\theta}\,d\theta+\int_{-\pi}^{\pi}ix\sin\theta\, e^{ix\sin\theta}\,d\theta$$

Next, we evaluate the first integral using the method of integration by parts:

$$\int_{-\pi}^{\pi}x^2\cos^2\theta\, e^{ix\sin\theta}\,d\theta=\int_{-\pi}^{\pi}(-ix\cos\theta)(ix\cos\theta)\,e^{ix\sin\theta}\,d\theta$$

$$=\cancelto{0}{(-ix\cos\theta)\,e^{ix\sin\theta}\Big|_{-\pi}^{\pi}}-\int_{-\pi}^{\pi}(ix\sin\theta)\,e^{ix\sin\theta}\,d\theta$$

This 1st term will then be cancelled by the second integral obtained earlier, thus confirming that Bessel's equation is satisfied by $f(x)$.

$$f(0)=\int_{-\pi}^{\pi}e^{i0\sin\theta}\,d\theta=\int_{-\pi}^{\pi}d\theta=2\pi$$

Therefore, $\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ix\sin\theta}\,d\theta$ is equal to 1 at $x=0$ and also satisfies Bessel's equation with $n=0$. We conclude, therefore, that $J_0(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ix\sin\theta}\,d\theta$. Note that, since the integrand is periodic with a period of 2π, the range of the integral could as well be any 2π interval.