Problem 12) a) $J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!\, m!} \Rightarrow J_0'(x) = \sum_{m=0} \frac{(-1)^m 2m\, x^{2m-1}}{2^{2m} m!\, m!}$

$$\Rightarrow J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m (x/2)^{2m-1}}{(m-1)!\, m!} \underset{m'=m-1}{=} \sum_{m'=0}^{\infty} \frac{(-1)^{m'+1} (x/2)^{2m'+1}}{m'!\,(m'+1)!} = -J_1(x)$$

b) $$\frac{d}{dx}\left[x^n J_n(x)\right] = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2n}}{2^{2m+n} m!\,(m+n)!} = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2n) x^{2m+2n-1}}{2^{2m+n} m!\,(m+n)!}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n-1}}{2^{2m+n-1} m!\,(m+n-1)!} = x^n \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n-1}}{m!\,(m+n-1)!} = x^n J_{n-1}(x)$$

$$\Rightarrow n x^{n-1} J_n(x) + x^n J_n'(x) = x^n J_{n-1}(x) \Rightarrow \frac{n}{x} J_n(x) + J_n'(x) = J_{n-1}(x).$$

Similarly, $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!} = \sum_{m=0}^{\infty} \frac{(-1)^m 2m\, x^{2m-1}}{2^{2m+n} m!\,(n+m)!}$$

$$= x^{-n} \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m+n-1}}{2^{2m+n-1} (m-1)!\,(n+m)!} = x^{-n} \sum_{m'=0}^{\infty} \frac{(-1)^{m'+1} (x/2)^{2m'+n+1}}{m'!\,(n+1+m')!} = -x^{-n} J_{n+1}(x).$$

$$\Rightarrow -n x^{-n-1} J_n(x) + x^{-n} J_n'(x) = -x^{-n} J_{n+1}(x) \Rightarrow \frac{n}{x} J_n(x) - J_n'(x) = J_{n+1}(x).$$

Subtracting the two equations thus obtained, we find

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right].$$

c) Similarly, adding the two equations yields:

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right].$$