Problem 10) $x^2 f''(x) + x f'(x) + (x^2 - \frac{1}{4}) f(x) = 0.$

$$f(x) = \sum_{k=0}^{\infty} A_k x^{k+s} \Rightarrow f'(x) = \sum_{k=0}^{\infty} (k+s) A_k x^{k+s-1} \Rightarrow f''(x) = \sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^{k+s-2}.$$

Substitution into the differential equation now yields:

$$\sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^{k+s} + \sum_{k=0}^{\infty} (k+s) A_k x^{k+s} + \sum_{k=0}^{\infty} A_k x^{k+s+2} - \frac{1}{4} \sum_{k=0}^{\infty} A_k x^{k+s} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} \left[(k+s)^2 - \tfrac{1}{4}\right] A_k x^{k+s} + \sum_{k=2}^{\infty} A_{k-2} x^{k+s} = 0 \Rightarrow$$

$$\begin{cases} k=0: & (s^2 - \frac{1}{4}) A_0 = 0 \Rightarrow s = \pm\frac{1}{2} \text{ or } A_0 = 0. \\ k=1: & [(1+s)^2 - \frac{1}{4}] A_1 = 0 \Rightarrow s = -\frac{1}{2}, -\frac{3}{2} \text{ or } A_1 = 0. \\ k \geq 2: & [(k+s)^2 - \frac{1}{4}] A_k + A_{k-2} = 0. \end{cases}$$

Let $s = -\frac{1}{2}$, then both A_0 and A_1 will be arbitrary. The recursion relation becomes: $[(k - \frac{1}{2})^2 - \frac{1}{4}] A_k + A_{k-2} = 0 \Rightarrow A_k = -\dfrac{A_{k-2}}{k(k-1)} \Rightarrow$

$$A_2 = -\frac{A_0}{1 \cdot 2}; \quad A_4 = -\frac{A_2}{3 \cdot 4} = \frac{A_0}{1 \cdot 2 \cdot 3 \cdot 4}; \quad \ldots \quad A_{2m} = \frac{(-1)^m A_0}{(2m)!}$$

$$A_3 = -\frac{A_1}{2 \cdot 3}; \quad A_5 = -\frac{A_3}{4 \cdot 5} = \frac{A_1}{2 \cdot 3 \cdot 4 \cdot 5}; \quad \ldots \quad A_{2m+1} = \frac{(-1)^m A_1}{(2m+1)!}$$

$$f_1(x) = x^{-1/2} \left\{ A_0 - \frac{A_0}{2!} x^2 + \frac{A_0}{4!} x^4 - \cdots + \frac{(-1)^m A_0}{(2m)!} x^{2m} + \cdots \right\} = A_0 \frac{\cos x}{\sqrt{x}}.$$

$$f_2(x) = x^{-1/2} \left\{ A_1 x - \frac{A_1}{3!} x^3 + \frac{A_1}{5!} x^5 - \cdots + \frac{(-1)^m A_1}{(2m+1)!} x^{2m+1} + \cdots \right\} = A_1 \frac{\sin x}{\sqrt{x}}.$$

The general solution of the equation is thus $f(x) = A_0 \dfrac{\cos x}{\sqrt{x}} + A_1 \dfrac{\sin x}{\sqrt{x}}$. Incidentally, the differential equation is Bessel's equation of order $m = \frac{1}{2}$, and $f_1(x)$ and $f_2(x)$ are its two independent solutions.

The other solutions of the indicial equations must yield the same answers as above. We now derive the solutions for $(s = +\frac{1}{2}, A_1 = 0)$ and $(s = -\frac{3}{2}, A_0 = 0)$.

Case of $s = +\frac{1}{2}$, $A_1 = 0$: Recursion relation $\left[(k+\frac{1}{2})^2 - \frac{1}{4}\right] A_k + A_{k-2} = 0 \Rightarrow$

$$A_k = -\frac{A_{k-2}}{k(k+1)} \Rightarrow$$

$$A_2 = -\frac{A_0}{3\cdot 2}; \qquad A_4 = -\frac{A_2}{4\cdot 5} = \frac{A_0}{5!}; \quad \cdots \quad A_{2m} = \frac{(-1)^m A_0}{(2m+1)!}$$

$$f_1(x) = x^{1/2}\left\{ A_0 - \frac{A_0}{3!}x^2 + \frac{A_0}{5!}x^4 - \cdots + \frac{(-1)^m A_0}{(2m+1)!}x^{2m} + \cdots \right\}$$

$$= A_0 x^{-1/2}\left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^m}{(2m+1)!}x^{2m+1} + \cdots \right\} = A_0 \frac{\sin x}{\sqrt{x}}.$$

Case of $s = -\frac{3}{2}$, $A_0 = 0$: Recursion relation $\left[(k-\frac{3}{2})^2 - \frac{1}{4}\right] A_k + A_{k-2} = 0 \Rightarrow$

$$(k^2 - 3k + 2)A_k = -A_{k-2} \Rightarrow A_k = -\frac{A_{k-2}}{(k-1)(k-2)} \Rightarrow$$

$$A_3 = -\frac{A_1}{1\cdot 2}; \qquad A_5 = -\frac{A_3}{3\cdot 4} = \frac{A_1}{4!}, \quad \cdots \quad A_{2m+1} = \frac{(-1)^m A_1}{(2m)!}$$

$$\Rightarrow f_2(x) = x^{-3/2}\left\{ A_1 x - \frac{A_1}{2!}x^3 + \frac{A_1}{4!}x^5 - \cdots + \frac{(-1)^m A_1}{(2m)!}x^{2m+1} + \cdots \right\}$$

$$= x^{-1/2}\left\{ A_1 - \frac{A_1}{2!}x^2 + \frac{A_1}{4!}x^4 - \cdots + \frac{(-1)^m A_1}{(2m)!}x^{2m} + \cdots \right\} = A_1 \frac{\cos x}{\sqrt{x}}.$$