

Problem 9) a)
$$z_1(t) = \int_{-\infty}^{\infty} Z_1(s) e^{+i2\pi st} ds \Rightarrow z_1'(t) = \int_{-\infty}^{\infty} i2\pi s Z_1(s) e^{+i2\pi st} ds$$

$$\Rightarrow z_1''(t) = \int_{-\infty}^{\infty} (i2\pi s)^2 Z_1(s) e^{+i2\pi st} ds.$$

Therefore, $F\{z_1(t)\} = Z_1(s)$; $F\{z_1'(t)\} = i2\pi s Z_1(s)$; $F\{z_1''(t)\} = -4\pi^2 s^2 Z_1(s)$.

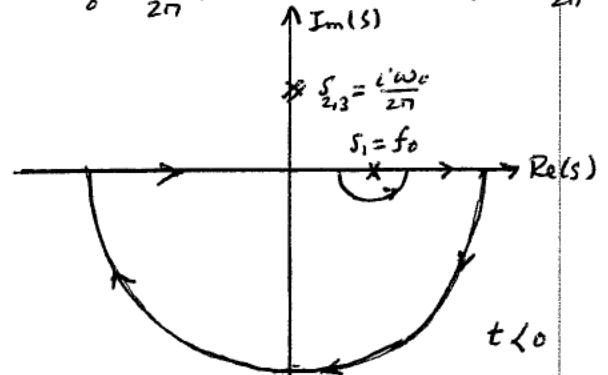
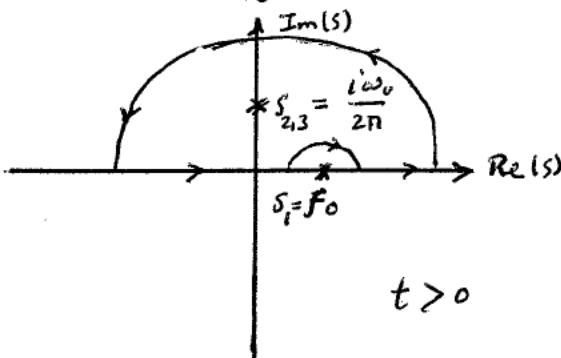
Since $f(t) = F_0 \text{step}(t) \sin(2\pi f_0 t) = \text{Im} \{ F_0 \text{step}(t) e^{i2\pi f_0 t} \}$, we solve the problem for the excitation function $F_0 \text{step}(t) e^{i2\pi f_0 t}$, then, in the end, take the imaginary part of $z_1(t)$.

$$F\{F_0 \text{step}(t) e^{i2\pi f_0 t}\} = F_0 \left[\frac{1}{2} \delta(s) - \frac{i}{2\pi s} \right] * \delta(s - f_0) = \frac{F_0}{2} \delta(s - f_0) - \frac{i F_0}{2\pi(s - f_0)}.$$

The F.T. of the differential equation thus yields:

$$Z_1(s) = \frac{\frac{F_0}{2m} \delta(s - f_0) - \frac{i F_0}{2\pi m(s - f_0)}}{-4\pi^2 s^2 + i4\pi s \omega_0 + \omega_0^2} = \frac{F_0 \delta(s - f_0)}{-8\pi^2 m \left(f_0 - \frac{i\omega_0}{2\pi}\right)^2} + \frac{i F_0}{8\pi^3 m (s - f_0) \left(s - \frac{i\omega_0}{2\pi}\right)^2}.$$

$$\text{Therefore, } z_1(t) = \int_{-\infty}^{\infty} Z_1(s) e^{+i2\pi st} ds = -\frac{F_0}{8\pi^2 m} \frac{e^{i2\pi f_0 t}}{\left(f_0 - \frac{i\omega_0}{2\pi}\right)^2} + \frac{i F_0}{8\pi^3 m} \int_{-\infty}^{\infty} \frac{e^{+i2\pi st}}{(s - f_0) \left(s - \frac{i\omega_0}{2\pi}\right)^2} ds.$$



For $t < 0$, Jordan's lemma dictates the use of the lower-half-plane. The only relevant pole here is $s_1 = f_0$ and, since it's on-axis, only half of the residue will contribute to the integral. The sign of the residue is negative because of the clockwise direction of travel on the small semi-circle.

We thus have:

$$t < 0 : z_1(t) = - \frac{F_0}{8\pi^2 m} \frac{e^{i2\pi f_0 t}}{(f_0 - \frac{i\omega_0}{2\pi})^2} - \frac{1}{2} (2\pi i) \frac{iF_0}{8\pi^3 m} \frac{e^{i2\pi f_0 t}}{(f_0 - \frac{i\omega_0}{2\pi})^2} = 0 \quad \checkmark$$

For $t > 0$, Jordan's Lemma dictates the use of the upper-half-plane.

Here the pole at $s_1 = f_0$ contributes one-half of its residue (because it is on-axis), but the sign of its contribution will be positive, because of the counter-clockwise direction of travel on the small semi-circle.

The contribution of the double-pole (or second-order pole) at $s = \frac{i\omega_0}{2\pi}$ is

given by the derivative of $\frac{e^{i2\pi s t}}{s - f_0}$ evaluated at $s = \frac{i\omega_0}{2\pi}$, that is,

$$\begin{aligned} \left. \frac{d}{ds} \left(\frac{e^{i2\pi s t}}{s - f_0} \right) \right|_{s = \frac{i\omega_0}{2\pi}} &= \frac{i2\pi t e^{i2\pi (\frac{i\omega_0}{2\pi}) t}}{\frac{i\omega_0}{2\pi} - f_0} - \frac{e^{i2\pi (\frac{i\omega_0}{2\pi}) t}}{(\frac{i\omega_0}{2\pi} - f_0)^2} \\ &= \frac{4\pi^2 t e^{-\omega_0 t}}{\omega_0 + i2\pi f_0} + \frac{4\pi^2 e^{-\omega_0 t}}{(\omega_0 + i2\pi f_0)^2} \end{aligned}$$

$$\text{Therefore, } z_1(t) = - \frac{F_0}{8\pi^2 m} \frac{e^{i2\pi f_0 t}}{(f_0 - \frac{i\omega_0}{2\pi})^2} + \frac{1}{2} (2\pi i) \frac{iF_0}{8\pi^3 m} \frac{e^{i2\pi f_0 t}}{(f_0 - \frac{i\omega_0}{2\pi})^2}$$

$$+ (2\pi i) \frac{iF_0}{8\pi^3 m} \left\{ \frac{4\pi^2 t e^{-\omega_0 t}}{(\omega_0 + i2\pi f_0)} + \frac{4\pi^2 e^{-\omega_0 t}}{(\omega_0 + i2\pi f_0)^2} \right\}$$

$$= \frac{F_0 e^{i2\pi f_0 t}}{m(\omega_0 + i2\pi f_0)^2} - \frac{F_0 e^{-\omega_0 t}}{m(\omega_0 + i2\pi f_0)^2} \left[1 + (\omega_0 + i2\pi f_0) t \right]$$

$$= \frac{F_0}{m(\omega_0^2 + 4\pi^2 f_0^2)} e^{2i \tan^{-1}(2\pi f_0/\omega_0)} \left[e^{i2\pi f_0 t} - e^{-\omega_0 t} - \sqrt{\omega_0^2 + 4\pi^2 f_0^2} e^{i \tan^{-1}(2\pi f_0/\omega_0)} \times t e^{-\omega_0 t} \right]$$

Note that $z_1(0^+) = 0$ and also $z_1'(0^+) = 0$, as required by the continuity conditions for a second-order differential equation.

Next, we take the imaginary part of $z_1(t)$ as the final solution, because the excitation function is the imaginary part of $F_0 \text{step}(t) e^{i2\pi f_0 t}$. We'll have:

$$z_1(t) = \frac{F_0 \text{step}(t)}{m(\omega_0^2 + 4\pi^2 f_0^2)} \left\{ \sin[2\pi f_0 t - 2 \tan^{-1}(2\pi f_0 / \omega_0)] + \sin[2 \tan^{-1}(2\pi f_0 / \omega_0)] e^{-\omega_0 t} + \sqrt{\omega_0^2 + 4\pi^2 f_0^2} \sin[\tan^{-1}(2\pi f_0 / \omega_0)] t e^{-\omega_0 t} \right\} \Rightarrow$$

$$z_1(t) = \frac{F_0 \text{step}(t)}{m(\omega_0^2 + 4\pi^2 f_0^2)} \left\{ \sin[2\pi f_0 t - 2 \tan^{-1}(\frac{2\pi f_0}{\omega_0})] + \frac{4\pi f_0 \omega_0 e^{-\omega_0 t}}{\omega_0^2 + 4\pi^2 f_0^2} + 2\pi f_0 t e^{-\omega_0 t} \right\}$$

b) Because of damping, the natural frequency ω_0 may be used for excitation without causing the excitation amplitude to go to infinity. Thus, when $2\pi f_0 = \omega_0$, all we need to do is substitute $2\pi f_0$ with ω_0 in the above expression. We'll have:

$$z_1(t) = \frac{F_0 \text{step}(t)}{2m\omega_0^2} \left\{ \sin(\omega_0 t - \frac{\pi}{2}) + \frac{2\omega_0^2 e^{-\omega_0 t}}{2\omega_0^2} + \omega_0 t e^{-\omega_0 t} \right\} \Rightarrow$$

$$z_1(t) = \frac{F_0 \text{step}(t)}{2m\omega_0^2} \left[(1 + \omega_0 t) e^{-\omega_0 t} - \cos(\omega_0 t) \right].$$

Once again, it is seen that $z_1(0^+) = z_1'(0^+) = 0$, as required by the continuity condition.