**Problem 7)** a) Let the Fourier transform of $f(x)$ be $F(s) = \int_{-\infty}^{\infty} f(x)\exp(-i2\pi sx)\,dx$. Then $f(x) = \int_{-\infty}^{\infty} F(s)\exp(i2\pi sx)\,ds$, and $\mathcal{F}\{f'(x)\} = i2\pi sF(s)$. The Fourier transform of the differential equation may thus be written as

$$i2\pi sF(s) + \eta F(s) = \text{Sinc}(s) \qquad \to \qquad F(s) = \frac{\sin(\pi s)}{\pi s\,(i2\pi s + \eta)}. \qquad (1)$$

The solution of the differential equation may now be obtained by inverse Fourier transforming the above $F(s)$, as follows:

$$f(x) = \mathcal{F}^{-1}\{F(s)\} = \int_{-\infty}^{\infty} \frac{\sin(\pi s)}{\pi s\,(i2\pi s + \eta)}\exp(i2\pi sx)\,ds$$

$$= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{\exp(i\pi s) - \exp(-i\pi s)}{s[s - i(\eta/2\pi)]}\exp(i2\pi sx)\,ds$$

$$= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{\exp[i2\pi(x+\frac{1}{2})s]}{s[s - i(\eta/2\pi)]}\,ds + \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{\exp[i2\pi(x-\frac{1}{2})s]}{s[s - i(\eta/2\pi)]}\,ds. \qquad (2)$$

The integrands on the right-hand-side of Eq.(2) have two poles, one at $z_0 = 0$, the other at $z_1 = i\eta/2\pi$, as shown in the figures below. Depending on the value of x, the integration contour may be in the upper- or lower-half of the complex plane. The contribution of the large semi-circle to the loop integral vanishes when its radius R goes to infinity (Jordan's lemma). As for the pole at z_0, only one-half of its residue must be taken into account because this pole is located directly on the x-axis.
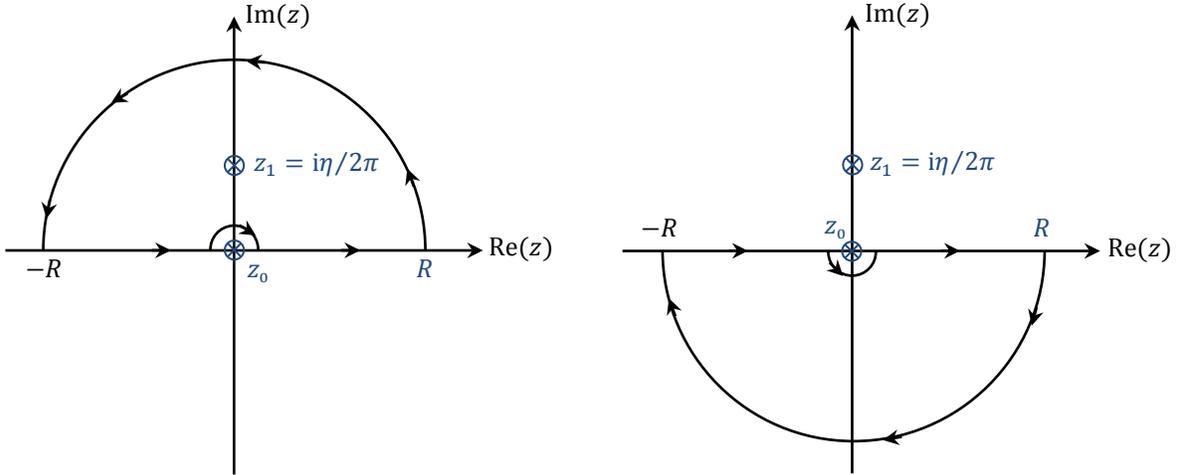
Both integrals must be evaluated in the lower-half of the complex plane when $x < -\frac{1}{2}$. Thus

$$\int_{-\infty}^{\infty} \frac{\exp[i2\pi(x\pm\frac{1}{2})s]}{s[s - i(\eta/2\pi)]}\,ds = -i\pi\frac{\exp[i2\pi(x\pm\frac{1}{2})z_0]}{z_0 - i(\eta/2\pi)} = \frac{2\pi^2}{\eta}. \qquad (3)$$

The same result continues to apply to the second integral for $x < \frac{1}{2}$ as well. If $x > -\frac{1}{2}$, the first integral must be evaluated in the upper-half plane, as follows:

$$\int_{-\infty}^{\infty} \frac{\exp[i2\pi(x+\frac{1}{2})s]}{s[s - i(\eta/2\pi)]}\,ds = i2\pi\frac{\exp[i2\pi(x+\frac{1}{2})z_1]}{z_1} + i\pi\frac{\exp[i2\pi(x+\frac{1}{2})z_0]}{z_0 - i(\eta/2\pi)}$$

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$$= (4\pi^2/\eta)\{\exp[-\eta(x + \tfrac{1}{2})] - \tfrac{1}{2}\}. \tag{4}$$

Finally, if $x > \tfrac{1}{2}$, the second integral must also be evaluated in the upper-half-plane, that is,

$$\int_{-\infty}^{\infty} \frac{\exp[i2\pi(x-\frac{1}{2})s]}{s[s - i(\eta/2\pi)]}\, ds = i2\pi \frac{\exp[i2\pi(x-\frac{1}{2})z_1]}{z_1} + i\pi \frac{\exp[i2\pi(x-\frac{1}{2})z_0]}{z_0 - i(\eta/2\pi)}$$

$$= (4\pi^2/\eta)\{\exp[-\eta(x - \tfrac{1}{2})] - \tfrac{1}{2}\}. \tag{5}$$

The complete solution is now obtained from Eq.(2) upon substitution from Eqs.(3)-(5), as follows:

$$f(x) = \begin{cases} 0; & x < -\tfrac{1}{2}, \\ \{1 - \exp[-\eta(x + \tfrac{1}{2})]\}/\eta; & -\tfrac{1}{2} < x < \tfrac{1}{2}, \\ [\exp(\eta/2) - \exp(-\eta/2)]\exp(-\eta x)/\eta; & x > \tfrac{1}{2}. \end{cases} \tag{6}$$

b) The function $f(x)$ is continuous at $x = -\tfrac{1}{2}$, where $f(x^-) = f(x^+) = 0$, and also at $x = \tfrac{1}{2}$, where $f(x^-) = f(x^+) = [1 - \exp(-\eta)]/\eta$. Any discontinuity in $f(x)$ would have been unacceptable, because the original differential equation contains $f'(x)$ on the left-hand side, but no corresponding delta-functions on the right-hand side. Note also that $f(x)$ approaches zero as $x \to \infty$, all of which in keeping with one's expectations from the solution of the differential equation.

c) A plot of $f(x)$ is shown below.