Problem 5) Legendre equation: $(1-x^2)f''(x) - 2xf'(x) + n(n+1)f(x) = 0$

Frobenius' Solution: $f(x) = \sum_{k=0}^{\infty} A_k x^{k+s} \Rightarrow f'(x) = \sum_{k=0}^{\infty} (k+s) A_k x^{k+s-1} \Rightarrow$

$f''(x) = \sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^{k+s-2}$. Placing these expressions into the Legendre equation yields:

$$\sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^{k+s-2} - \sum_{k=0}^{\infty} (k+s)(k+s-1) \overset{}{x^{k+s}} \underset{\hat{}}{A_k} - \sum_{k=0}^{\infty} 2(k+s) A_k x^{k+s}$$

$$+ \sum_{k=0}^{\infty} n(n+1) A_k x^{k+s} = 0$$

The first sum in the above equation may be written as follows:

$$\sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^{k+s-2} = s(s-1)A_0 x^{s-2} + (1+s)sA_1 x^{s-1} + \sum_{k=2}^{\infty} (k+s)(k+s-1) A_k x^{k+s-2}$$

$$= s(s-1)A_0 x^{s-2} + s(s+1)A_1 x^{s-1} + \sum_{k'=0}^{\infty} (k'+2+s)(k'+2+s-1) A_{k'+2} x^{k'+s} \leftarrow k' = k-2$$

$$= s(s-1)A_0 x^{s-2} + s(s+1)A_1 x^{s-1} + \sum_{k=0}^{\infty} (k+s+2)(k+s+1) A_{k+2} x^{k+s} \leftarrow \text{Dummy } k' \to k$$

Combining the above results, we'll have:

$s(s-1)A_0 = 0$

$s(s+1)A_1 = 0$

$(k+s+2)(k+s+1)A_{k+2} = [(k+s)(k+s-1) + 2(k+s) - n(n+1)]A_k \leftarrow k = 0, 1, 2, \cdots$

If we choose $s=0$ as the solution of the indicial equations, both $A_0$ and $A_1$ will become arbitrary parameters. The recursion relation then yields the values of the remaining coefficients as follows:

$$A_{k+2} = \frac{k(k+1)-n(n+1)}{(k+1)(k+2)} A_k . \qquad k = 0, 1, 2, 3, \cdots$$

If we choose $s=1$ as the solution, then $A_0$ will be arbitrary, but $A_1$ must be zero. The recursion relation will yield $A_3 = A_5 = A_7 = \cdots = 0$, and $A_{k+2} = \frac{(k+1)(k+2)-n(n+1)}{(k+2)(k+3)} A_k$. This is similar to the previous recursion except that $k$ has been incremented by 1. Considering that for $s=0$, $f(x) = \sum_{k=0}^{\infty} A_k x^k$, whereas for $s=1$, $f(x) = \sum_{k=0}^{\infty} A_k x^{k+1} = \sum_{k=1}^{\infty} A_{k-1} x^k$

Clearly the index $k$ is shifted by 1 unit; therefore, the solution obtained for $s=1$ is inherent in the solution obtained for $s=0$.

Similarly, if we choose $s=-1$ as the solution of the indicial equations, then $A_0 = 0$, and $A_1$ will be arbitrary. The recursion relation ensures that $A_2 = A_4 = A_6 = \cdots = 0$, and that $A_{k+2} = \frac{(k-1)k - n(n+1)}{k(k+1)} A_k$ for $k = 1, 3, 5, 7, \cdots$.

This is again similar to the recursion obtained for $s=0$, except that $k$ has been decremented by 1. We note, however that, for $s=-1$, we'll have $f(x) = \sum_{k=0}^{\infty} A_k x^{k-1} = \sum_{k=0}^{\infty} A_{k+1} x^k$ ($A_0$ is already known to be zero).

Clearly the index $k$ is shifted by one unit (relative to the case of $s=0$).

Therefore, the solution obtained for $s=-1$ is also inherent in the solution for $s=0$.

Returning now to the general solution obtained for $s=0$, we try to write the numerator of the recursion relation as a product of two terms. The 2nd-order polynomial $k(k+1)-n(n+1)$ has two roots:

$$k^2+k-n(n+1)=0 \Rightarrow k=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+n(n+1)} = -\frac{1}{2}\pm\sqrt{(n+\frac{1}{2})^2} = n, -(n+1).$$

Therefore, $k(k+1)-n(n+1)=(k-n)(k+n+1)$. We'll have:

$$A_{k+2} = \frac{(k-n)(k+n+1)}{(k+1)(k+2)} A_k \Rightarrow$$

$$A_2 = -\frac{n(n+1)}{1.2} A_0 \; ; \quad A_4 = \frac{(2-n)(3+n)}{3\cdot 4}\,\frac{-n(n+1)}{1.2} A_0 = \frac{A_0}{4!}\, n(n-2) \times (n+1)(n+3) \; ;$$

$$A_6 = -\frac{A_0}{6!} n(n-2)(n-4) \times (n+1)(n+3)(n+5) \; ; \quad \ldots$$

$$A_{2m} = \frac{(-1)^m A_0}{(2m)!} n(n-2)(n-4)\cdots(n-2m+2) \times (n+1)(n+3)\cdots(n+2m-1)$$

Note that, if $n$ is an even integer, the above series terminates at $k=2m=n$, whereas for odd-integer values of $n$, the series continues indefinitely.

Similarly, $A_3 = -\frac{(n-1)(n+2)}{2\cdot 3} A_1$ ; $A_5 = \frac{(n-3)(n+4)}{4\cdot 5}\,\frac{(n-1)(n+2)}{2\cdot 3} A_1 = \frac{A_1}{5!}(n-1)(n-3) \times (n+2)(n+4)$

$$\ldots \quad A_{2m+1} = \frac{(-1)^m A_1}{(2m+1)!}(n-1)(n-3)\cdots(n-2m+1) \times (n+2)(n+4)\cdots(n+2m)$$

Note that, if $n$ is an odd integer, the above series terminates at $k=2m+1=n$, whereas for even-integer values of $n$, the series continues indefinitely.

The two independent solutions of the Legendre equation are thus given by:

$$f_1(x) = A_0 + \sum_{m=1}^{\infty} A_{2m} x^{2m} \qquad f_2(x) = A_1 x + \sum_{m=1}^{\infty} A_{2m+1} x^{2m+1}$$