**Problem 4**) a) The current flowing through the circuit is related to the voltage across the capacitor via $i(t) = C\dot{v}_c(t)$. According to Kirchhoff's voltage law, the sum of the resistor voltage $Ri(t)$ and the capacitor voltage $v_c(t)$ must equal the applied voltage $v(t)$. Therefore,

$$RC\dot{v}_c(t) + v_c(t) = v(t) \qquad \rightarrow \qquad \tau(t)\dot{v}_c(t) + v_c(t) = v(t). \tag{1}$$

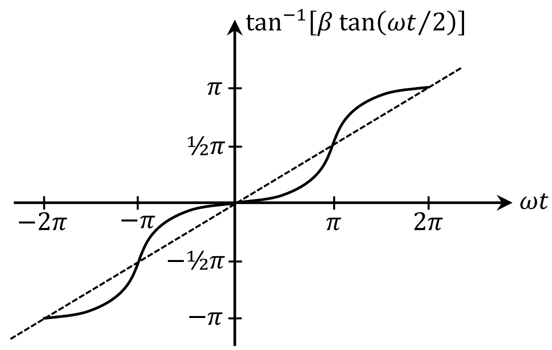
b) The above differential equation is similar to the 1st order ODE that has been analyzed in Sec.2. Using Eq.(3), Sec.2, we find

$$g(t) = [1/\tau(t)]\exp[\textstyle\int \mathrm{d}t/\tau(t)] = \frac{1}{\tau_0[1+\alpha\cos(\omega t)]}\exp\left(\frac{1}{\tau_0}\int_{-\infty}^{t}\frac{\mathrm{d}t'}{1+\alpha\cos(\omega t')}\right)$$
$$= \frac{1}{\tau_0[1+\alpha\cos(\omega t)]}\exp\left(\frac{1}{\tau_0\omega}\int_{-\infty}^{\omega t}\frac{\mathrm{d}\theta}{1+\alpha\cos\theta}\right). \tag{2}$$

The integral appearing in the exponent on the right-hand side of Eq.(2) was evaluated in Chapter 3, Section 2, Example 6. We thus have

$$g(t) = \frac{1}{\tau_0[1+\alpha\cos(\omega t)]}\exp\left(\frac{2\tan^{-1}\left[\sqrt{(1-\alpha)/(1+\alpha)}\tan(\omega t/2)\right]}{\sqrt{1-\alpha^2}\,\tau_0\omega}\right). \tag{3}$$

When $\alpha = 0$, Eq.(3) reduces to $g(t) = (1/\tau_0)\exp(t/\tau_0)$, in agreement with the results obtained in Example 3, Sec.2. When $0 < |\alpha| < 1$, the coefficient of $\tan(\omega t/2)$ will be some positive number other than 1. Here one must be careful, because the arctangent function has $\pm\pi$ discontinuities whenever $\omega t$ passes through an odd-multiple of $\pi$. Nevertheless, once the angles have been unwrapped, the exponent of $g(t)$ becomes a periodically wobbling function of $t$, such as that shown on the right. The exponent rises continually with time as $t/(\sqrt{1-\alpha^2}\tau_0)$, albeit with a superposed periodic wobble whose frequency is $\omega$ and whose amplitude depends on $\alpha$, being greater for larger values of $|\alpha|$. The subsequent division of the exponential function by $\tau_0[1+\alpha\cos(\omega t)]$, as required by Eq.(3), further modulates the exponential function by multiplying it with a periodic function of time whose frequency also happens to be $\omega$.




Substitution of the function $g(t)$ given by Eq.(3) above into Eq.(4), Sec.2, finally yields

$$v_c(t) = \int_{-\infty}^{t}\exp\left(-\frac{2\tan^{-1}\left[\sqrt{(1-\alpha)/(1+\alpha)}\tan(\omega t/2)\right]}{\sqrt{1-\alpha^2}\,\tau_0\omega}\right)g(t')v(t')\mathrm{d}t'. \tag{4}$$

Aside from wiggles of frequency $\omega$ that the integrand in the above equation has inherited from the oscillations of the circuit parameters, the final result in Eq.(4) is qualitatively similar to that for an $RC$ circuit having a time-invariant $\tau = RC$, as given by Eq.(9) in Example 3, Sec.2.