Problem 3) A trial Solution for this type of equation is $f(x) = e^{\alpha x}$:

$$\frac{d^2}{dx^2} e^{\alpha x} + a_1 \frac{d}{dx} e^{\alpha x} + a_0 e^{\alpha x} = 0 \Rightarrow \alpha^2 + a_1 \alpha + a_0 = 0 \Rightarrow$$

$$\alpha_{1,2} = -\frac{1}{2} a_1 \pm \sqrt{\frac{a_1^2}{4} - a_0}$$

Case I: $\frac{a_1^2}{4} > a_0$, that is, $a_1 > 2\sqrt{a_0}$. In this case both α_1 and α_2 are real and negative. The general solution of the differential equation is then given by $f(x) = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$, where A ad B are arbitrary constants. For the solution to be real-valued, both A and B must be real.

Case II: $a_1 < 2\sqrt{a_0}$. In this case we write $\alpha_{1,2} = -\frac{a_1}{2} \pm i\sqrt{a_0 - \frac{a_1^2}{4}}$.

The general solution is then given by $f(x) = Ae^{(-\frac{a_1}{2} + i\sqrt{a_0 - a_1^2/4})x} + Be^{(-\frac{a_1}{2} - i\sqrt{a_0 - a_1^2/4})x}$. Since the two exponentials are conjugates of each other, for $f(x)$ to be real-valued, we must have $B = A^*$. Writing $A = |A|e^{i\phi}$ and $B = |A|e^{-i\phi}$ we'll have:

$$f(x) = |A|e^{i\phi} e^{-\frac{1}{2}a_1 x} e^{i\sqrt{a_0 - a_1^2/4}\,x} + |A|e^{-i\phi} e^{-\frac{1}{2}a_1 x} e^{-i\sqrt{a_0 - a_1^2/4}\,x} \Rightarrow$$

$$f(x) = 2|A|e^{-\frac{1}{2}a_1 x} \cos\left(\sqrt{a_0 - \frac{a_1^2}{4}}\,x + \phi\right).$$

The real-valued constants $|A|$ ad ϕ are arbitrary parameters.

Case III: $a_1 = 2\sqrt{a_0}$. In this case $\alpha_1 = \alpha_2$, and we only have one solution in the form of $e^{\alpha x}$. To find a second solution, we employ a limit process in which we begin by assuming that $\alpha_1 \neq \alpha_2$, then allow $\alpha_1 \to \alpha_2$.

If $e^{\alpha_1 x}$ and $e^{\alpha_2 x}$ are both solutions of the differential equation, their sum and diffence will also be solutions, that is, $f_1(x) = \frac{1}{2}(e^{\alpha_1 x} + e^{\alpha_2 x})$ and $f_2(x) = \frac{1}{2}(e^{\alpha_1 x} - e^{\alpha_2 x})$. When $\alpha_1 \to \alpha_2$ we'll have $f_1(x) \to e^{\alpha_1 x}$. However, $f_2(x)$ approaches zero. To find the limiting form of $f_2(x)$ we write:

$$f_2(x) = \frac{1}{2}\left[e^{\alpha_1 x} - e^{(\alpha_1 + \alpha_2 - \alpha_1)x}\right] = \frac{1}{2} e^{\alpha_1 x}\left[1 - e^{(\alpha_2 - \alpha_1)x}\right] = \frac{1}{2} e^{\alpha_1 x}\left[\cancel{1} - \cancel{1} - (\alpha_2 - \alpha_1)x - \cdots\right]$$

$$\to \frac{-1}{2}(\alpha_2 - \alpha_1)\, x\, e^{\alpha_1 x}$$

when $\alpha_1 \to \alpha_2$. The coefficient $\frac{1}{2}(\alpha_1 - \alpha_2)$ is a constant (assuming that α_1 and α_2 are brought sufficiently close to each other, then fixed). We conclude that the two solutions of the differential equation (when $\alpha_1 = \alpha_2 = \alpha$) are $f_1(x) = e^{\alpha x}$ and $f_2(x) = x e^{\alpha x}$. The second solution can be readily verified by putting it into the differential equation. The general solution is thus given by

$$f(x) = (A + Bx)\, e^{-\frac{1}{2} a_1 x}.$$

For the above $f(x)$ to be real-valued, both A and B must be real constants.