Problem 2) For a capacitor $Q = CV \Rightarrow \frac{dQ}{dt} = C\frac{dV}{dt} \Rightarrow I = C\frac{dV}{dt}$.

For a resistor $V = RI$. Therefore, $V(t) = RI(t) + V_c(t) \Rightarrow$

$$RC\frac{dV_c(t)}{dt} + V_c(t) = V(t).$$

a) Because $\text{Step}(t) = 0$ from $t = -\infty$ to 0, the capacitor is empty during the time interval $t < 0$. Therefore, we need not worry about the initial voltage of the capacitor. Fourier transforming the differential equation we find: $RC(i2\pi s)\underbrace{\underset{\sim}{V}_c(s)}_{\text{F.T. of } V_c(t)} + \underset{\sim}{V}_c(s) = \mathcal{F}\{V(t)\} \Rightarrow$

$$\underset{\sim}{V}_c(s) = \frac{V_0\,\mathcal{F}\{\text{Step}(t)\}}{1 + i2\pi RCs} = \frac{\frac{1}{2}V_0\delta(s) - i\frac{V_0}{2\pi s}}{1 + i2\pi RCs} = \frac{V_0\delta(s)}{2(1 + i2\pi RCs)} - i\frac{V_0}{2\pi s(1 + i2\pi RCs)}$$

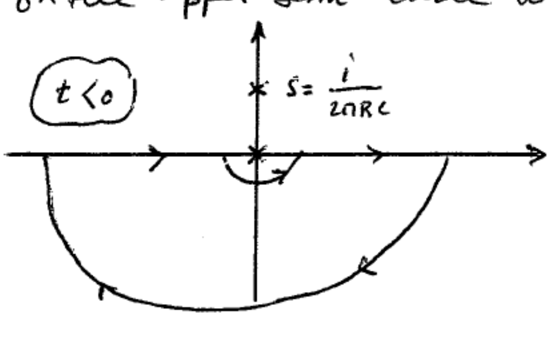
The function that multiplies $\delta(s)$ may be replaced by its value at $s = 0$. We thus have:

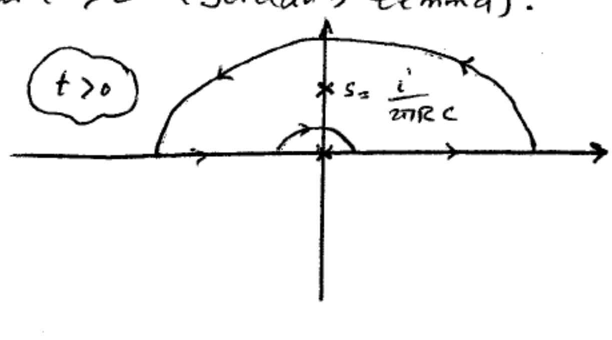
$$\underset{\sim}{V}_c(s) = \frac{1}{2}V_0\delta(s) - \frac{V_0}{4\pi^2 RC}\,\frac{1}{s\left(s - \frac{i}{2\pi RC}\right)}.$$

$$V_c(t) = \mathcal{F}^{-1}\{\underset{\sim}{V}_c(s)\} = \int_{-\infty}^{\infty}\underset{\sim}{V}_c(s)\,e^{+i2\pi st}\,ds = \frac{1}{2}V_0\int_{-\infty}^{\infty}\delta(s)e^{i2\pi st}\,ds - \frac{V_0}{4\pi^2 RC}\int_{-\infty}^{\infty}\frac{e^{i2\pi st}\,ds}{s\left(s - \frac{i}{2\pi RC}\right)}$$

$$= \frac{1}{2}V_0 - \frac{V_0}{4\pi^2 RC}\int_{-\infty}^{\infty}\frac{e^{i2\pi st}}{s\left(s - \frac{i}{2\pi RC}\right)}\,ds.$$

The integral must be evaluated on the lower semi-circle when $t < 0$, and on the upper semi-circle when $t > 0$ (Jordan's lemma).





For $t<0$ the integral will be minus one-half of the residue at $s=0$. That is,

$$\int_{-\infty}^{\infty} \frac{e^{i2\pi st}}{s(s-\frac{i}{2\pi RC})}\,ds = -\frac{1}{2}(2\pi i)\frac{1}{0-\frac{i}{2\pi RC}} = 2\pi^2 RC \quad \leftarrow t<0$$

because $s=0$ is on-axis

Therefore, $V_c(t) = \frac{1}{2}V_0 - \frac{V_0}{4\pi^2 RC}(2\pi^2 RC) = \frac{1}{2}V_0 - \frac{1}{2}V_0 = 0 \quad \leftarrow t<0$

For $t>0$, the integral will be equal to one-half of the residue at $s=0$, plus the residue at $s=\frac{i}{2\pi RC}$, that is,

$$\int_{-\infty}^{\infty} \frac{e^{i2\pi st}}{s(s-\frac{i}{2\pi RC})}\,ds = 2\pi i\left\{\frac{1}{2}\,\frac{1}{0-\frac{i}{2\pi RC}} + \frac{e^{-t/RC}}{i/2\pi RC}\right\}$$

$$= -2\pi^2 RC + 4\pi^2 RC\, e^{-t/RC} \quad \leftarrow t \geq 0$$

Therefore, $V_c(t) = \frac{1}{2}V_0 - \frac{V_0}{4\pi^2 RC}\left[-2\pi^2 RC + 4\pi^2 RC\, e^{-t/RC}\right] \Rightarrow$

$$V_c(t) = V_0(1-e^{-t/RC}) \quad \leftarrow t \geq 0$$

Combining the above results we'll have $V_c(t) = V_0(1-e^{-t/RC})\,\mathrm{step}(t)$.

b) When $V(t) = \delta(t)$ we'll have $\mathcal{F}\{V(t)\} = 1$. Therefore,

$$\underset{\sim}{V}_c(s) = \frac{1}{1+i2\pi RCs} \Rightarrow V_c(t) = \mathcal{F}^{-1}\{\underset{\sim}{V}_c(s)\} = \int_{-\infty}^{\infty}\frac{e^{i2\pi st}}{1+i2\pi RCs}\,ds$$

$$= \frac{1}{i2\pi RC}\int_{-\infty}^{\infty}\frac{e^{i2\pi st}}{s-\frac{i}{2\pi RC}}\,ds.$$

The only pole here is at $s=\frac{i}{2\pi RC}$. For $t<0$ the integral must be done on the semi-circle in the lower half-plane (to satisfy Jordan's lemma). But here there are no poles and, therefore, the integral will be zero. For $t>0$, the integral is done over the semi-circle in the upper-half-plane, where the only enclosed pole is $s=\frac{i}{2\pi RC}$. Consequently,

$$V_c(t)=\frac{1}{i2\pi RC}(2\pi i)\frac{e^{-t/RC}}{1}=+\frac{1}{RC}e^{-t/RC}\ ;\ t>0.$$

Combining the above results, we'll have $V_c(t)=\frac{1}{RC}e^{-t/RC}\,\text{Step}(t).$

Alternatively, since $\delta(t)$ is the derivative of Step(t), the impulse-response could be found directly by differentiating the step-response (setting $V_0=1$). Therefore,

$$\text{Impulse response}=\frac{d}{dt}\left[(1-e^{-t/RC})\,\text{Step}(t)\right]=\frac{1}{RC}e^{-t/RC}\,\text{Step}(t)+(1-e^{-t/RC})\,\delta(t).$$

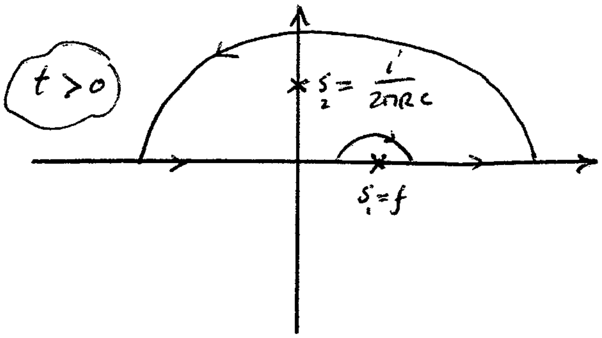
The coefficient of $\delta(t)$ in the above expression is equal to zero at $t=0$. Therefore, the last term drops out, and what's left is identical with the result obtained previously.

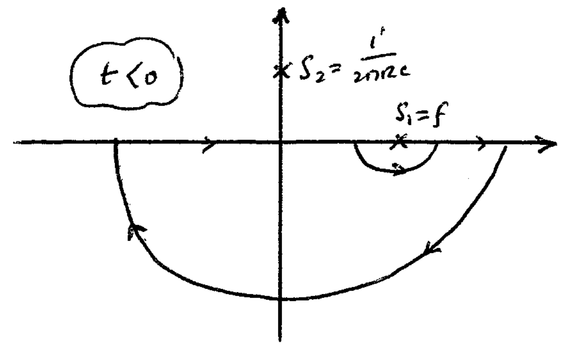
c) $\mathcal{F}\{V(t)\}=\mathcal{F}\{V_0e^{i2\pi ft}\,\text{Step}(t)\}=V_0\delta(s-f)*\{\frac{1}{2}\delta(s)-\frac{i}{2\pi s}\}$

$$=\frac{1}{2}V_0\delta(s-f)-\frac{iV_0}{2\pi(s-f)}.$$

$$\underset{\sim}{V}_c(s)=\frac{\frac{1}{2}V_0\delta(s-f)-\frac{iV_0}{2\pi(s-f)}}{1+i2\pi RCs}=\frac{\frac{1}{2}V_0}{1+i2\pi RCf}\delta(s-f)-\frac{V_0}{4\pi^2RC}\frac{1}{(s-f)(s-\frac{i}{2\pi RC})}$$

Therefore, $V_c(t) = \int_{-\infty}^{\infty} \frac{V_0}{2(1+i2\pi RCf)} \delta(s-f) e^{+i2\pi st} ds - \frac{V_0}{4\pi^2 RC} \int_{-\infty}^{\infty} \frac{e^{+i2\pi st}}{(s-f)(s-\frac{i}{2\pi RC})} ds.$





For $t<0$, the second integral is equal to minus one-half of the residue at $s_1 = f$, that is,

$$V_c(t) = \frac{V_0 e^{i2\pi ft}}{2(1+i2\pi RCf)} + \frac{V_0}{4\pi^2 RC} \frac{1}{2}(2\pi i) \frac{e^{i2\pi ft}}{f - \frac{i}{2\pi RC}}$$

$$= \frac{V_0 e^{i2\pi ft}}{2(1+i2\pi RCf)} - \frac{V_0 e^{i2\pi ft}}{2(1+i2\pi RCf)} = 0 \leftarrow t<0$$

for $t>0$, the second integral is taken over the upper semi-circle. Its value will be given by one-half the residue at $s_1 = f$ plus the residue at $S_2 = \frac{i}{2\pi RC}$. We'll have:

$$V_c(t) = \frac{V_0 e^{i2\pi ft}}{2(1+i2\pi RCf)} - \frac{V_0}{4\pi^2 RC}(2\pi i)\left\{\frac{e^{i2\pi ft}}{2(f-\frac{i}{2\pi RC})} + \frac{e^{-t/RC}}{\frac{i}{2\pi RC} - f}\right\}$$

$$= \frac{V_0 e^{i2\pi ft}}{1+i2\pi RCf} - V_0 \frac{e^{-t/RC}}{1+i2\pi RCf} = \frac{V_0}{1+i2\pi RCf}\left(e^{i2\pi ft} - e^{-t/RC}\right) \leftarrow t>0$$

Note that $V_c(0^+) = 0$, which is a required boundary condition for the first-order differential equation that governs the evolution of $V_c(t)$.

The above results may now be combined to yield:

$$V_c(t) = \frac{V_0}{1+i2\pi RCf}\left(e^{i2\pi ft} - e^{-t/RC}\right) \text{step}(t).$$