Problem 1)

$$f(x) = \int_{-\infty}^{\infty} F(s)\, e^{+i2\pi s x}\, ds \;\Rightarrow\; f'(x) = \int_{-\infty}^{\infty} (i2\pi s)\, F(s)\, e^{i2\pi s x}\, ds \;\Rightarrow$$

$$\frac{d^n}{dx^n} f(x) = \int_{-\infty}^{\infty} (i2\pi s)^n\, F(s)\, e^{i2\pi s x}\, ds$$

Fourier transforming both sides of the differential equation yields:

$$a_n (i2\pi s)^n F(s) + a_{n-1} (i2\pi s)^{n-1} F(s) + \cdots + a_1 (i2\pi s) F(s) + a_0 F(s) = G(s)$$

$$\Rightarrow F(s) = \frac{G(s)}{a_0 + a_1 (i2\pi s) + \cdots + a_{n-1} (i2\pi s)^{n-1} + a_n (i2\pi s)^n}$$

Now, when $g(x) = \delta(x)$ we will have $G(s) = 1$. The impulse response will be denoted by $h(x)$, and its Fourier Transform by $H(s)$. Thus

$$H(s) = \frac{1}{a_0 + a_1 (i2\pi s) + \cdots + a_{n-1} (i2\pi s)^{n-1} + a_n (i2\pi s)^n}$$

Clearly, for the arbitrary input $g(x)$, we will have $F(s) = G(s) H(s)$. Using the convolution theorem this yields $f(x) = g(x) * h(x)$.

Note that the independent variable x ranges from $-\infty$ to $+\infty$ in this problem. Therefore, all the initial conditions and boundary conditions are already built into the solution $f(x) = g(x) * h(x)$.