Problem 42) a, b) Both Fourier integrals can be evaluated by first completing the square in the exponent of the integrand, then switching the real variable x to the complex variable z — with the aid of the contours shown in the figure below. We will have

$$\mathcal{F}\{\exp(-\alpha x^2)\exp(\pm \mathrm{i}2\pi s_0 x)\}$$

$$= \int_{-\infty}^{\infty} \exp(-\alpha x^2)\exp(\pm \mathrm{i}2\pi s_0 x)\exp(-\mathrm{i}2\pi s x)\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \exp[-\alpha x^2 - \mathrm{i}2\pi(s \mp s_0)x]\,\mathrm{d}x$$

Completing the square →

$$= \int_{-\infty}^{\infty} \exp\{-\alpha[x + \mathrm{i}\pi\alpha^{-1}(s \mp s_0)]^2 - \pi^2\alpha^{-1}(s \mp s_0)^2\}\,\mathrm{d}x$$

Use contour in Fig.(a) when $(s \mp s_0) > 0$; use contour in Fig.(b) when $(s \mp s_0) < 0$. Integrand has no poles inside the contour. Vertical legs do *not* contribute in the limit when $L \to \infty$. →

$$= \exp\{-[\pi(s \mp s_0)/\sqrt{\alpha}]^2\} \int_{-\infty}^{\infty} \exp\{-\alpha[\underbrace{x + \mathrm{i}\pi\alpha^{-1}(s \mp s_0)}_{z}]^2\}\,\underbrace{\mathrm{d}x}_{\mathrm{d}z}$$

$$= \exp\{-[\pi(s \mp s_0)/\sqrt{\alpha}]^2\} \underbrace{\int_{-\infty}^{\infty} \exp(-\alpha x^2)\,\mathrm{d}x}$$

$$= \sqrt{\pi/\alpha}\,\exp\left\{-\pi\left[\sqrt{\pi/\alpha}\,(s \mp s_0)\right]^2\right\}. \qquad (1)$$

← $= \sqrt{\pi/\alpha}$; see below

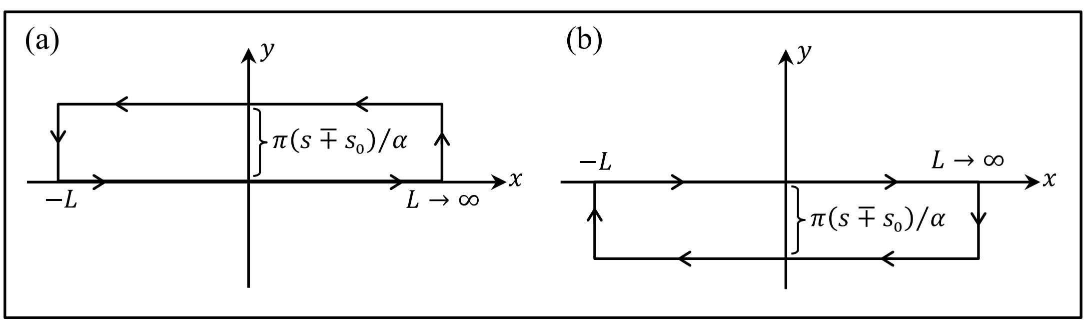
The integral $\int_{-\infty}^{\infty} \exp(-\alpha x^2)\,\mathrm{d}x$ appearing in the penultimate line of Eq.(1) is evaluated as follows:

$$\left[\int_{-\infty}^{\infty} \exp(-\alpha x^2)\,\mathrm{d}x\right]^2 = \iint_{-\infty}^{\infty} \exp[-\alpha(x^2 + y^2)]\,\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{\infty} \exp(-\alpha r^2)\,2\pi r\mathrm{d}r = (\pi/\alpha)\int_0^{\infty} 2x\exp(-x^2)\,\mathrm{d}x = \pi/\alpha. \qquad (2)$$

Alternatively, the change of variable $x = \sqrt{\pi/\alpha}\,y$ yields

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\,\mathrm{d}x = \sqrt{\pi/\alpha}\int_{-\infty}^{\infty} \exp(-\pi y^2)\,\mathrm{d}y = \sqrt{\pi/\alpha}. \qquad (3)$$



c) Returning to the final expression in Eq.(1), the Gaussian function $\exp(-\pi s^2)$ is even and has area equal to 1. Scaling the variable s by the constant coefficient $\sqrt{\pi/\alpha}$ narrows the function and reduces its area to $\sqrt{\alpha/\pi}$. However, multiplying the function by $\sqrt{\pi/\alpha}$ restores the area under the function to 1. The argument of the function being $s \mp s_0$ indicates that the center of the Gaussian has shifted to $s = \pm s_0$. We thus have two tall, narrow, and symmetric functions, each having an area equal to 1, one centered at $s = s_0$, the other at $s = -s_0$. In the limit when $\alpha \to 0$, these become $\delta(s - s_0)$ and $\delta(s + s_0)$, respectively. Considering that $f(x) = \cos(2\pi s_0 x) = ½[\exp(\mathrm{i}2\pi s_0 x) + \exp(-\mathrm{i}2\pi s_0 x)]$, we conclude that $F(s) = ½[\delta(s - s_0) + \delta(s + s_0)]$.