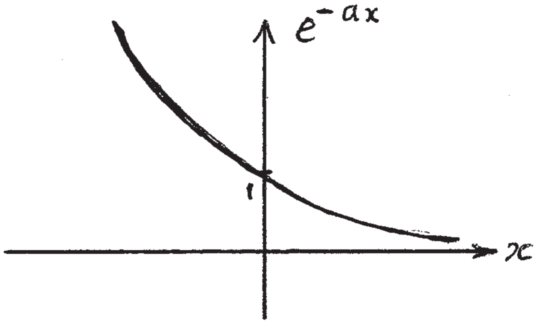
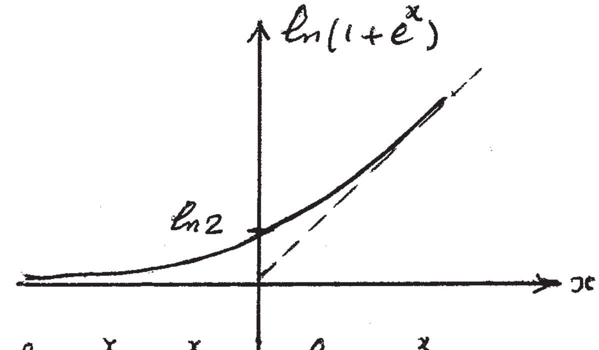
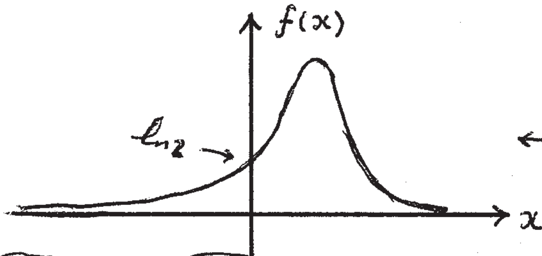
Problem 41)





$\ln(1+e^x) \to e^x$ as $x \to -\infty$ $\quad$ $\ln(1+e^x) \to x$ as $x \to \infty$



← $f(x) = e^{-ax}\ln(1+e^x)$ approaches $e^{(1-a)x}$ when $x \to -\infty$. It approaches xe^{-ax} when $x \to \infty$. The function peaks somewhere to the right of the origin, depending on the value of a.

When a is complex, i.e., $a = a' + ia''$, the above function will be multiplied by $e^{-ia''x}$.

On the small circle, $z = i\pi + \epsilon e^{i\theta}$. Therefore,

$$\oint_{\text{small circle}} f(z)e^{-i2\pi sz}dz = \int_{\theta=0}^{2\pi} e^{-a(i\pi+\epsilon e^{i\theta})}\ln(1+e^{i\pi+\epsilon e^{i\theta}})\,e^{-i2\pi s(i\pi+\epsilon e^{i\theta})}\,i\epsilon e^{i\theta}d\theta$$

$$= i\epsilon e^{-i\pi a}e^{2\pi^2 s}\int_0^{2\pi}\underbrace{e^{-(a+i2\pi s)\epsilon e^{i\theta}}}_{\simeq 1}\ln(1-\underbrace{e^{\epsilon e^{i\theta}}}_{})\,e^{i\theta}d\theta$$

$$\ln(1-e^{\epsilon e^{i\theta}}) \simeq -\epsilon e^{i\theta}$$

$$\simeq i e^{2\pi^2 s - i\pi a}\int_0^{2\pi}\epsilon[\ln\epsilon + i(\pi+\theta)]e^{i\theta}d\theta \to 0 \quad \text{as } \epsilon \to 0.$$

Note how the logarithmic function, $\ln(1+e^z)$ changes as z goes around the small circle. With $z = i\pi + \epsilon e^{i\theta}$ we have:

$$\ln(1+e^z) = \ln(1-e^{\epsilon e^{i\theta}}) \simeq \ln(1-1-\epsilon e^{i\theta}) = \ln\epsilon + i(\pi+\theta).$$

As θ goes from 0 to 2π (counterclockwise around the circle), the log function is incremented by $2\pi i$.

On the upper segment of the branch-cut: $\int_0^\infty e^{-a(x+i\pi)} \ln(1-e^{x}) e^{-i2\pi s(x+i\pi)} dx.$

On the lower segment of the branch-cut: $-\int_0^\infty e^{-a(x+i\pi)} [2\pi i + \ln(1-e^{x})] e^{-i2\pi s(x+i\pi)} dx.$

Adding the above contributions to the contour integral causes $\ln(1-e^{x})$ to cancel out, leaving: $-2\pi i\, e^{\pi(2\pi s - ia)} \int_0^\infty e^{-(a+i2\pi s)x} dx = \dfrac{-2\pi i\, e^{-i\pi(a+i2\pi s)}}{a+i2\pi s}.$

On the upper leg of the rectangular contour, where $z = x + 2\pi i$, we have:

$$-\int_{-\infty}^{\infty} e^{-a(x+2\pi i)} \ln(1+e^{x+2\pi i}) e^{-i2\pi s(x+2\pi i)} dx$$

$$= -e^{-i2\pi(a+i2\pi s)} \int_{-\infty}^{\infty} e^{-ax} \ln(1+e^{x}) e^{-i2\pi s x} dx = -e^{-i2\pi(a+i2\pi s)} F(s).$$

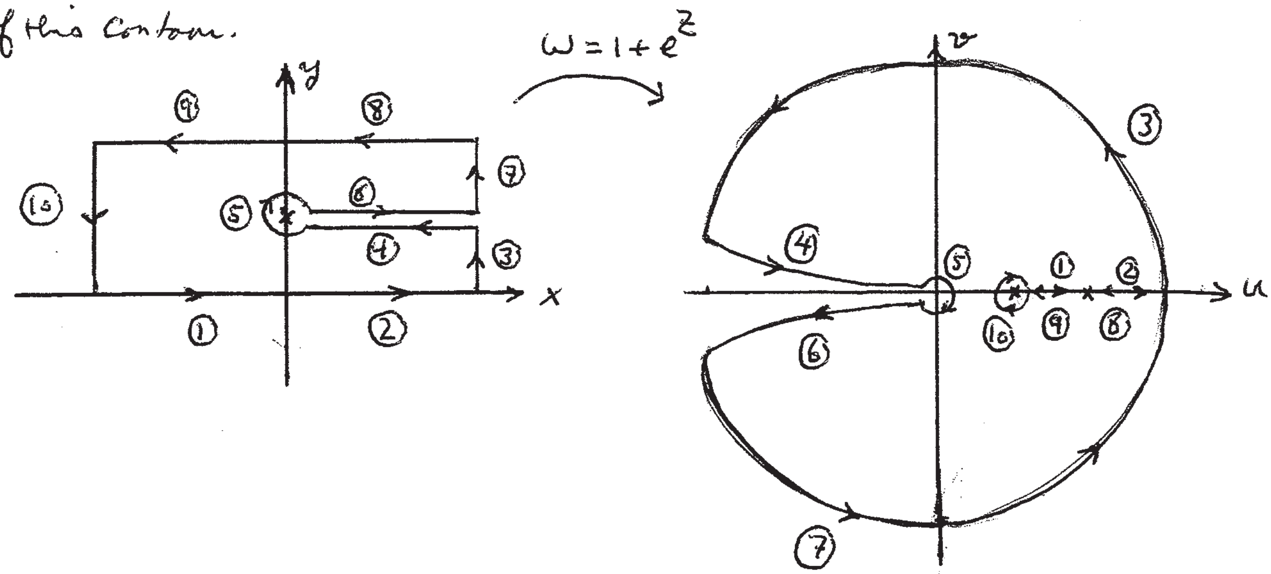
Adding all the contributions to the contour integral and using Cauchy's theorem, we'll have:

$$F(s) - e^{-i2\pi(a+i2\pi s)} F(s) - 2\pi i \frac{e^{-i\pi(a+i2\pi s)}}{a+i2\pi s} = 0$$

$$\Rightarrow F(s) = +2\pi i \frac{e^{-i\pi(a+i2\pi s)}}{(a+i2\pi s)(1-e^{-i2\pi(a+i2\pi s)})} = \frac{\pi}{(a+i2\pi s)\sin[\pi(a+i2\pi s)]}.$$ ✓

Note that the area under the function $f(x)$ is $F(0) = \dfrac{\pi}{a\sin(\pi a)}$.

Further discussion of problem 41) A map of the integration contour from the z-plane onto the ω-plane, where $\omega = 1+e^z$, shows the significance of this contour.



On the branch-cut, ④ and ⑥ correspond to $z = x + i(\pi - \epsilon)$ and $z = x + i(\pi+\epsilon)$, respectively. The two exponential curves labeled ④ and ⑥ in the ω-plane will approach the negative u-axis when $\epsilon \to 0$. On the branch-cut in the ω-plane, the value of $\log\omega$ on the upper segment ④ is larger than that on the lower segment ⑥ by $2\pi i$. This diagram fully justifies the values assigned to $\log(1+e^z)$ in the solution to problem 41.