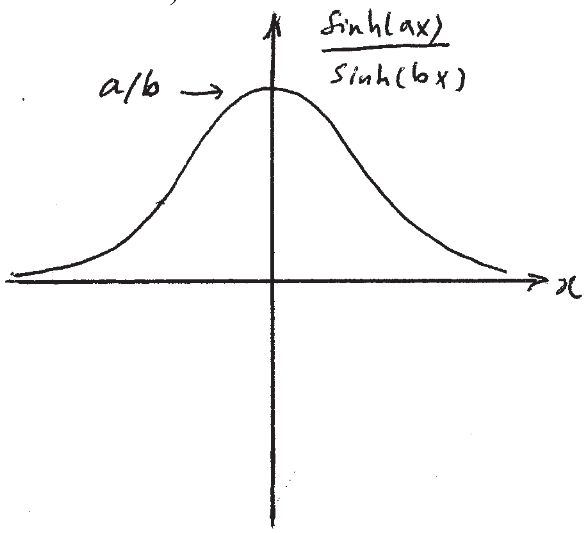
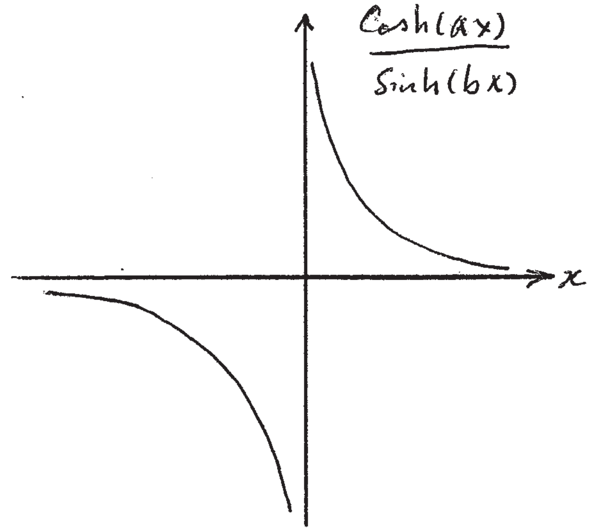
Problem 40)





On the large semi-circle the integral of $\dfrac{e^{\pm az}}{\sinh(bz)} e^{-i2\pi sz}$ goes to zero as $R \to \infty$ (Jordan's lemma). (Note that for $s<0$ the semi-circle must be in the upper half-plane, whereas for $s>0$, the semi-circle must be in the lower half-plane.) We choose to work with $s<0$ here. The results can be readily shown to be valid for $s>0$ as well, by simply repeating the same calculations in the lower half-plane.

$$\text{Residue at } z_n = \frac{e^{\pm a z_n}}{(-1)^n b} e^{-i2\pi s z_n} = \frac{(-1)^n}{b} e^{(2\pi^2 ns \pm i\pi na)/b}$$

$$= \frac{1}{b}\left[-e^{\frac{2\pi^2 s \pm i\pi a}{b}}\right]^n \qquad n = 0, 1, 2, \dots$$

We can now add all the residues using the geometric series formula, $\sum_{n=0}^{\infty} \alpha^n = \dfrac{1}{1-\alpha}$, where $|\alpha|<1$ is valid, of course, because $s<0$ has been assumed.

$$\text{Therefore, Sum of all residues} = \frac{1}{b\left[1+\exp\left(\frac{2\pi^2 s \pm i\pi a}{b}\right)\right]}.$$

The pole at $z_0 = 0$ requires special attention, because it sits on the x-axis, along which the integrals need to be evaluated. The small semi-circular path included in the contour allows $z_0 = 0$ to fall within the contour, but now we must consider the contribution of the small semi-circular path itself.

$$\oint_{\substack{\text{Small}\\ \text{Semi-circle}}} \frac{e^{\pm a \epsilon e^{i\theta}}}{b \cancel{\epsilon e^{i\theta}}} e^{-i2\pi s \epsilon e^{i\theta}} i \cancel{\epsilon e^{i\theta}} d\theta \underset{\epsilon \to 0}{=} \frac{i}{b} \int_{\theta=\pi}^{2\pi} d\theta = \frac{i\pi}{b} .$$

We may now apply Cauchy's theorem to write:

$$\int_{-\infty}^{\infty} \frac{e^{\pm a x}}{\sinh(bx)} e^{-i2\pi s x} dx + \frac{i\pi}{b} = \frac{2\pi i}{b\left[1 + \exp\left(\frac{2\pi^2 s \pm i\pi a}{b}\right)\right]} .$$

Therefore,

$$F(s) = \mathcal{F}\left\{\frac{\sinh(ax)}{\sinh(bx)}\right\} = \frac{1}{2}\mathcal{F}\left\{\frac{e^{ax}}{\sinh(bx)}\right\} - \frac{1}{2}\mathcal{F}\left\{\frac{e^{-ax}}{\sinh(bx)}\right\}$$

$$= \frac{i\pi}{b}\left\{\frac{1}{1+\exp\left(\frac{2\pi^2 s + i\pi a}{b}\right)} - \frac{1}{1+\exp\left(\frac{2\pi^2 s - i\pi a}{b}\right)}\right\}$$

$$= \frac{i\pi}{b} \frac{-2i \exp(2\pi^2 s/b)\sin(\pi a/b)}{1 + \exp(4\pi^2 s/b) + 2\exp(2\pi^2 s/b)\cos(\pi a/b)}$$

$$= \frac{\pi}{b} \frac{2\sin(\pi a/b)}{\exp(-2\pi^2 s/b) + \exp(2\pi^2 s/b) + 2\cos(\pi a/b)} = \frac{(\pi/b)\sin(\pi a/b)}{\cos(\pi a/b) + \cosh(2\pi^2 s/b)} .$$

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$$G(s) = \mathcal{F}\left\{\frac{\cosh(ax)}{\sinh(bx)}\right\} = \frac{1}{2}\mathcal{F}\left\{\frac{e^{ax}}{\sinh(bx)}\right\} + \frac{1}{2}\mathcal{F}\left\{\frac{e^{-ax}}{\sinh(bx)}\right\}$$

$$= \frac{i\pi}{b}\left\{\frac{2 + 2\exp(2\pi^2 s/b)\cos(\pi a/b)}{1 + \exp(4\pi^2 s/b) + 2\exp(2\pi^2 s/b)\cos(\pi a/b)} - 1\right\}$$

$$= \frac{i\pi}{b} \frac{1 - \exp(4\pi^2 s/b)}{1 + \exp(4\pi^2 s/b) + 2\exp(2\pi^2 s/b)\cos(\pi a/b)} = \frac{-i(\pi/b)\sinh(2\pi^2 s/b)}{\cos(\pi a/b) + \cosh(2\pi^2 s/b)} .$$

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