Problem 39) We apply the method of stationary-phase with $f(x) = \exp(-x^2)$ and $g(x) = \sqrt{1 - x^2}$ to this integral. The stationary point is found by setting $g'(x) = -x(1 - x^2)^{-\frac{1}{2}} = 0$, from which the sole stationary point is found to be $x_0 = 0$. This is an acceptable stationary point as it resides within the $(-1, 1)$ integration range. At the stationary point, the value of $f(x)$ is 1, that of $g(x)$ is also 1, and the value of the second derivative $g''(x) = -(1 - x^2)^{-3/2}$ at $x = x_0$ is found to be -1. We have shown in the class that the approximate value of the integral is

$$I \cong \sqrt{\frac{2\pi}{|\eta g''(x_0)|}} f(x_0) \exp[i\eta g(x_0)] \exp(\pm i\pi/4); \quad \pm \text{ depending on whether } \eta g''(x_0) \gtrless 0.$$

Substitution for the various parameters in the above expression thus yields

$$I \cong \sqrt{2\pi/\eta} \exp[i(\eta - \pi/4)].$$

It is instructive to numerically evaluate the integral for large values of η to see how close the approximate value comes to the actual value of the integral. A *Matlab®* program for such a calculation is listed below.

```
%
% Evaluating the Stationary-Phase Approximation to an Integral
%
clear all
eta=25;
nmax=1000;
deltax=2/nmax;
%
I=0;
for n=1:nmax
    x=-1+(n-1)*deltax;
    x2=x*x;
    I=I+exp(-x2+i*eta*sqrt(1-x2));
end
Integral=I*deltax
%
Approximation=sqrt(2*pi/eta)*exp(i*(eta-pi/4))
%
%End of Program
```