

Problem 38)

The stationary-phase approximation applies to

$$I = \int_a^b f(x) e^{i\eta g(x)} dx. \text{ At each stationary-point of } g(x), \text{ i.e., a}$$

point such as x_0 where $g'(x_0) = 0$, the contribution of that stationary

$$\text{point to the integral is given by } \sqrt{\frac{2\pi}{\eta |g''(x_0)|}} f(x_0) e^{i\eta g(x_0)} e^{\pm i\pi/4},$$

with \pm sign depending on whether $g''(x_0)$ is > 0 or < 0 .

In the present problem $J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin \theta} d\theta$. Thus x plays the

role of η , which becomes large when the asymptotic behavior of

$J_0(x)$ is desired. Since $g(\theta) = \sin \theta$, we'll have $g'(\theta) = \cos \theta = 0$
 $\Rightarrow \theta_0 = \pi/2$ and $\theta_1 = 3\pi/2$, so there are two stationary points.
 We must then calculate the contribution of each stationary point to the integral as follows:

$$\theta_0 = \pi/2; \quad g(\theta_0) = \sin(\pi/2) = 1; \quad g''(\theta_0) = -\sin(\pi/2) = -1 \Rightarrow$$

$$\text{Contribution of } \theta_0 \text{ to the integral: } \sqrt{\frac{2\pi}{x|-1|}} e^{ix} e^{-i\pi/4} = \sqrt{\frac{2\pi}{x}} e^{i(x-\pi/4)}$$

$$\theta_1 = \frac{3\pi}{2}; \quad g(\theta_1) = \sin(\frac{3\pi}{2}) = -1; \quad g''(\theta_1) = -\sin(\frac{3\pi}{2}) = +1 \Rightarrow$$

$$\text{Contribution of } \theta_1 \text{ to the integral: } \sqrt{\frac{2\pi}{x|+1|}} e^{-ix} e^{+i\pi/4} = \sqrt{\frac{2\pi}{x}} e^{-i(x-\pi/4)}$$

$$\text{Consequently, } J_0(x) \sim \frac{1}{2\pi} \left\{ \sqrt{\frac{2\pi}{x}} e^{i(x-\pi/4)} + \sqrt{\frac{2\pi}{x}} e^{-i(x-\pi/4)} \right\} = \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4).$$