Problem 37) a) $F(s, z=0^+) = \mathcal{F}\{f(x, z=0^+)\} = \int_{-\infty}^{\infty} f(x, z=0^+) e^{-i2\pi s x} dx$

Therefore, $f(x, z=0^+) = \mathcal{F}^{-1}\{F(s, z=0^+)\} = \int_{-\infty}^{\infty} F(s, z=0^+) e^{+i2\pi s x} ds$.

The complex-amplitude of a plane-wave propagating along the unit-vector $\vec{\sigma}$ in the $z>0$ half-space is $a(x, z) = a_0 \exp[i \frac{2\pi}{\lambda}(x\sigma_x + z\sigma_z)]$.

At $z=0^+$, the plane-wave's amplitude is given by $a(x, z=0^+) = a_0 e^{i\frac{2\pi}{\lambda} x \sigma_x}$.

Comparing this plane-wave amplitude with the distribution at $z=0^+$, we find that $s = \sigma_x/\lambda$ and $a_0 = F(s=\sigma_x/\lambda, z=0^+)$.

b) Since $\vec{\sigma}$ is a unit-vector, we have $\sigma_x^2 + \sigma_z^2 = 1 \Rightarrow \sigma_z = \sqrt{1-\sigma_x^2}$. In the paraxial regime where the scalar diffraction theory is applicable, the spatial spectrum $F(s, z=0^+)$ approaches zero at high spatial frequencies s, meaning that the significant values of s are those which satisfy $|s| \ll \frac{1}{\lambda}$. Therefore, plane-waves that constitute the transmitted beam in the $z>0$ half-space correspond to values of $\sigma_x \ll 1$. This means that the relevant values of σ_z in the present problem are all real. (When $\sigma_x > 1$, we'll have imaginary values for σ_z; these correspond to <u>evanescent</u> plane-waves.) In any event, at $z = z_0$ we must put the z-dependent term in the Fourier spectrum determined in part (a). We'll have: $f(x, z=z_0) = \int_{-\infty}^{\infty} F(s, z=0^+) e^{i2\pi(sx + z_0\sqrt{1-\lambda^2 s^2}/\lambda)} ds$.

The distribution at $z = z_0$ is thus obtained by first Fourier-Transforming $f(x, z=0^+)$, then multiplying this F.T. by $\exp[i2\pi z_0\sqrt{1-\lambda^2 s^2}/\lambda]$, and finally doing an inverse F.T. on the product function. This may be done analytically if the functions involved are simple. In practice, however, the initial distribution $f(x, z=0^+)$ is not a simple function, and the F.T. as well as the inverse F.T. operations must be done numerically. The function $\exp[i2\pi z_0\sqrt{1-\lambda^2 s^2}/\lambda]$ is easy to evaluate on a computer, provided that z_0 is not too large. As z_0 becomes larger, this exponential function oscillates rapidly with increasing s, making numerical calculations more and more difficult. It is under such circumstances that the stationary-phase approx. becomes useful.

c) $$f(x, z=z_0) = \int_{-\infty}^{\infty} F(s, z=0^+)\, e^{i2\pi z_0\left(\frac{x}{z_0}s + \frac{\sqrt{1-\lambda^2 s^2}}{\lambda}\right)} ds.$$

The above integral may be evaluated using the method of stationary phase, provided that $\eta = 2\pi z_0$ is sufficiently large. To determine the stationary-point, let $G(s) = (x/z_0)s + \frac{1}{\lambda}\sqrt{1-\lambda^2 s^2}$. Then

$$\frac{dG(s)}{ds} = \frac{x}{z_0} + \frac{1}{\lambda}\,\frac{-2\lambda^2 s}{2\sqrt{1-\lambda^2 s^2}} = 0 \Rightarrow \frac{\lambda s}{\sqrt{1-\lambda^2 s^2}} = \frac{x}{z_0} \Rightarrow (\lambda s)^2 = (x/z_0)^2(1-\lambda^2 s^2)$$

$$\Rightarrow \lambda s = \frac{x/z_0}{\sqrt{1+(x/z_0)^2}}.$$ ← This is the only solution that satisfies the condition for stationarity, i.e., $G'(s)=0$.

Next, we obtain the second derivative of $G(s)$ at the stationary-point:

$$G''(s) = \frac{d^2 G(s)}{ds^2} = \frac{d}{ds}\left(\frac{-\lambda s}{\sqrt{1-\lambda^2 s^2}}\right) = \frac{-\lambda}{\sqrt{1-\lambda^2 s^2}} - \lambda s\left(-\tfrac{1}{2}\right)(-2\lambda^2 s)(1-\lambda^2 s^2)^{-3/2}$$

$$= \frac{-\lambda(1-\lambda^2 s^2) - \lambda^3 s^2}{(1-\lambda^2 s^2)^{3/2}} = -\frac{\lambda}{(1-\lambda^2 s^2)^{3/2}}.$$

Evaluating $G''(s)$ at the stationary point yields:

$$G''(s)\Big|_{\substack{\text{stationary}\\ \text{point}}} = -\lambda\left[1-\frac{(x/z_0)^2}{1+(x/z_0)^2}\right]^{-3/2} = -\frac{\lambda}{[1+(x/z_0)^2]^{-3/2}} = -\lambda\left[1+\left(\frac{x}{z_0}\right)^2\right]^{3/2}$$

We have shown that $\int_a^b f(x)e^{i\eta g(x)}dx \simeq \sqrt{\frac{2\pi}{|\eta g''(x_0)|}}\, f(x_0)\, e^{i\eta g(x_0)}\, e^{\pm i\pi/4}$

with the $\pm$ sign depending on the sign of $\eta g''(x_0)$. Here $\eta = 2\pi z_0$ and

$g'' = -\lambda\left[1+\left(\frac{x}{z_0}\right)^2\right]^{3/2}$. Therefore,

$$f(x, z=z_0) \simeq \sqrt{\frac{\cancel{2\pi}}{\cancel{2\pi} z_0\lambda[1+(x/z_0)^2]^{3/2}}}\; F\left(\frac{x/z_0}{\lambda\sqrt{1+(x/z_0)^2}},\, z=0^+\right)$$

$$\times \exp\left\{i2\pi z_0\left[\frac{(x/z_0)^2}{\lambda\sqrt{1+(x/z_0)^2}} + \frac{1}{\lambda}\sqrt{1-\frac{(x/z_0)^2}{1+(x/z_0)^2}}\,\right]\right\} e^{-i\pi/4}$$

$$\Rightarrow f(x, z=z_0) \simeq \frac{1}{\sqrt{i\lambda z_0}}\left(\frac{z_0^2}{z_0^2+x^2}\right)^{3/4} F\left(\frac{x}{\lambda\sqrt{z_0^2+x^2}},\, z=0^+\right) e^{i2\pi(z_0/\lambda)\sqrt{1+(x/z_0)^2}}$$

The distance from the origin at $(x, z) = (0,0)$ to the point (x, z_0) in the far field is $r = \sqrt{x^2+z_0^2}$. Also, the angle subtended at the origin by the observation point at (x, z_0) is $\theta = \sin^{-1}\left(\frac{x}{\sqrt{z_0^2+x^2}}\right)$. Therefore,

$$f(x, z=z_0) = \frac{e^{i2\pi r/\lambda}}{\sqrt{i\lambda z_0}} \cos^{3/2}\theta\; F\left(\frac{\sin\theta}{\lambda}, z=0^+\right).$$

It is in the above sense that the far-field distribution is related to the Fourier transform of the light-amplitude distribution in the object-plane.

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