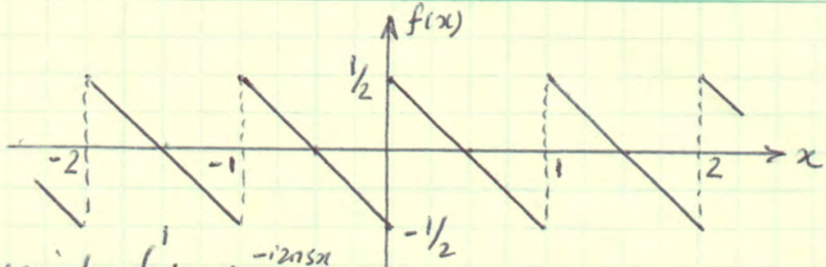
Problem 35)



$F(s) =$

Fourier transform of one period $= \int_0^1 (\tfrac{1}{2} - x) e^{-i2\pi sx} dx$

$$= \frac{1}{2}\,\frac{1}{-i2\pi s} e^{-i2\pi sx}\Big|_0^1 - \int_0^1 x e^{-i2\pi sx} dx = \frac{i}{4\pi s}\left(e^{-i2\pi s} - 1\right) - \frac{x}{-i2\pi s} e^{-i2\pi sx}\Big|_0^1 + \frac{1}{-i2\pi s}\int_0^1 e^{-i2\pi sx} dx$$

$$= \frac{e^{-i\pi s}\sin(\pi s)}{2\pi s} - \frac{i e^{-i2\pi s}}{2\pi s} - \frac{e^{-i2\pi s} - 1}{4\pi^2 s^2}$$

The Fourier series coefficients are obtained by sampling the above function at $s = \frac{n}{P} = n$, where $n = 0, \pm 1, \pm 2, \dots$. Therefore,

$$F(n) = \frac{e^{-i\pi n}\,\mathrm{sinc}(n)}{2} - \frac{i e^{-i2\pi n}}{2\pi n} - \frac{e^{-i2\pi n} - 1}{4\pi^2 n^2} = -\frac{i}{2\pi n} \quad \text{when } n \neq 0.$$

To find $F(0)$ we expand the second and third terms of $F(s)$ in a Taylor series, as follows:

$$\lim_{s \to 0} F(s) = \frac{1}{2} e^{-i\pi s}\,\mathrm{sinc}(s) - \frac{i(1 - i2\pi s + \cdots)}{2\pi s} - \frac{(\cancel{1} - i2\pi s - 2\pi^2 s^2 + \cdots) - \cancel{1}}{4\pi^2 s^2}$$

$$= \underbrace{\frac{1}{2} e^{-i\pi s}\,\mathrm{sinc}(s)} - \cancel{\frac{i}{2\pi s}} - 1 + O(s) + \cancel{\frac{i}{2\pi s}} + \frac{1}{2} + O(s)$$

$$= \frac{1}{2} - 1 + \frac{1}{2} + O(s) = O(s) \to 0 \text{ as } s \to 0. \Rightarrow F(0) = 0.$$

The Fourier series of $f(x)$ is thus given by:

$$f(x) = \sum_{n=-\infty}^{\infty} F(n) e^{i2\pi nx} = \sum_{n=1}^{\infty}\left(\frac{-i}{2\pi n} e^{i2\pi nx} + \frac{i}{2\pi n} e^{-i2\pi nx}\right) = \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n}$$

From the plot of $f(x)$ shown above it is clear that $f'(x) = -1 + \mathrm{comb}(x)$.

Thus $f'(x) = -1 + \mathrm{comb}(x) = \sum_{n=1}^{\infty} 2\cos(2\pi nx) \Rightarrow \mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} e^{i2\pi nx} \Rightarrow \mathcal{F}\{\mathrm{comb}(x)\} = \mathrm{comb}(s).$