Problem 34)

a) $\int_0^x \psi_{2s-1}(y)\,dy = \int_0^x \sum_{n=1}^{\infty} \frac{\cos(ny)}{n^{2s-1}}\,dy = \sum_{n=1}^{\infty} \frac{\int_0^x \cos(ny)\,dy}{n^{2s-1}}$

$= \sum_{n=1}^{\infty} \frac{\sin(ny)\big|_0^x}{n^{2s}} = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{2s}} = \psi_{2s}(x).$ ✓

b) $\int_0^x \psi_{2s}(y)\,dy = \int_0^x \sum_{n=1}^{\infty} \frac{\sin(ny)}{n^{2s}}\,dy = \sum_{n=1}^{\infty} \frac{\int_0^x \sin(ny)\,dy}{n^{2s}} = \sum_{n=1}^{\infty} \frac{-\cos(ny)\big|_0^x}{n^{2s+1}}$

$= -\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{2s+1}} + \sum_{n=1}^{\infty} \frac{1}{n^{2s+1}} = -\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{2s+1}} + \zeta(2s+1).$

Therefore, $\psi_{2s+1}(x) = \zeta(2s+1) - \int_0^x \psi_{2s}(y)\,dy.$ ✓