**Problem 27)** a) $\underset{\sim}{f}(x) = \begin{cases} f(x), & a \leqslant x < a+p; \\ 0, & x < a \text{ or } x \geqslant a+p. \end{cases}$

$$\underset{\sim}{F}(s) = \mathcal{F}\{\underset{\sim}{f}(x)\} = \int_{-\infty}^{\infty} \underset{\sim}{f}(x) e^{-i2\pi s x} dx = \int_{a}^{a+p} f(x) e^{-i2\pi s x} dx.$$

Now, $f(x) = \sum_{n=-\infty}^{\infty} \underset{\sim}{f}(x-np) = \underset{\sim}{f}(x) * \frac{1}{p}\,\text{Comb}(\frac{x}{p})$. Therefore,

$$F(s) = \mathcal{F}\{f(x)\} = \underset{\sim}{F}(s)\,\text{Comb}(ps) = \underset{\sim}{F}(s) \sum_{n=-\infty}^{\infty} \delta(ps-n) = \underset{\sim}{F}(s) \sum_{n=-\infty}^{\infty} \frac{1}{p}\delta(s-\frac{n}{p})$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{p} \underset{\sim}{F}(\frac{n}{p})\, \delta(s-\frac{n}{p})$$ ← for each δ-function, only the value of $\underset{\sim}{F}(s)$ at the location of δ-function matters.

Having found the F.T. of $f(x)$, we now write $f(x)$ as the inverse F.T. of $F(s)$:

$$f(x) = \mathcal{F}^{-1}\{F(s)\} = \int_{-\infty}^{\infty} F(s) e^{+i2\pi s x} ds = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{p} \underset{\sim}{F}(\frac{n}{p}) \delta(s-\frac{n}{p}) e^{+i2\pi s x} ds$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{p} \underset{\sim}{F}(\frac{n}{p}) \int_{-\infty}^{\infty} \delta(s-\frac{n}{p}) e^{+i2\pi s x} ds \underset{\text{sifting property}}{=} \sum_{n=-\infty}^{\infty} \frac{1}{p} \underset{\sim}{F}(\frac{n}{p}) e^{i2\pi n x/p}$$

Clearly, $c_n = \frac{1}{p} \underset{\sim}{F}(\frac{n}{p}) = \frac{1}{p} \int_{a}^{a+p} f(x) e^{-i2\pi n x/p} dx.$

b) Set $n=0$ to obtain $c_0 = \frac{1}{p} \int_{a}^{a+p} f(x) dx$. Here $f(x)$ could be real or complex.

c) If $f(x)$ is real-valued, we'll have $f^*(x) = f(x)$. Therefore,

$$c_{-n} = \frac{1}{p} \int_{a}^{a+p} f(x) e^{+i2\pi n x/p} dx = \left[ \frac{1}{p} \int_{a}^{a+p} f^*(x) e^{-i2\pi n x/p} dx \right]^* = c_n^*.$$

Then $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n x/p} = c_0 + \sum_{n=1}^{\infty} (c_n e^{i2\pi n x/p} + c_{-n} e^{-i2\pi n x/p})$.

When $f(x)$ is real, $C_{-n} = C_n^*$. Therefore, if $C_n = a_n - i b_n$, we'll have $C_{-n} = a_n + i b_n$. The above equation may then be written:

$$f(x) = C_0 + \sum_{n=1}^{\infty} (a_n - i b_n) e^{i 2\pi n x/P} + \sum_{n=1}^{\infty} (a_n + i b_n) e^{-i 2\pi n x/P} \Rightarrow$$

$$f(x) = C_0 + 2\sum_{n=1}^{\infty} a_n \cos(2\pi n x/P) + 2\sum_{n=1}^{\infty} b_n \sin(2\pi n x/P).$$

In the above equation $C_n = a_n - i b_n = \frac{1}{P}\int_a^{a+P} f(x) e^{-i 2\pi n x/P} dx$

$$= \frac{1}{P}\int_a^{a+P} f(x)\left[\cos(2\pi n x/P) - i \sin(2\pi n x/P)\right] dx \Rightarrow$$

$$a_n = \frac{1}{P}\int_a^{a+P} f(x) \cos(2\pi n x/P)\, dx$$ and $$b_n = \frac{1}{P}\int_a^{a+P} f(x) \sin(2\pi n x/P)\, dx.$$