Problem 26) a) $f(x) = \dfrac{2a}{a^2+(2\pi x)^2}\,\text{Comb}(x)$

We have shown in the class that $\mathcal{F}\left\{\dfrac{2a}{a^2+(2\pi x)^2}\right\} = e^{-a|s|}$.

Therefore $F(s) = e^{-a|s|} * \text{Comb}(s) = \sum_{n=-\infty}^{\infty} \delta(s-n) * e^{-a|s|} = \sum_{n=-\infty}^{\infty} e^{-a|s-n|}$

$$\Rightarrow F(0) = \sum_{n=-\infty}^{\infty} e^{-a|n|} = 1 + 2\sum_{n=1}^{\infty} e^{-an} = 1 + 2\,\frac{e^{-a}}{1-e^{-a}} = \frac{e^a+1}{e^a-1}.$$

geometric Series

Now, $\int_{-\infty}^{\infty} f(x)\,dx = F(0) \Rightarrow \int_{-\infty}^{\infty} \dfrac{2a}{a^2+(2\pi x)^2}\,\text{Comb}(x)\,dx = \dfrac{e^a+1}{e^a-1} \Rightarrow$

$$\sum_{n=-\infty}^{\infty} \frac{2a}{a^2+4\pi^2n^2} = \frac{2a}{a^2} + 2\sum_{n=1}^{\infty} \frac{2a}{a^2+4\pi^2n^2} = \frac{e^a+1}{e^a-1} \Rightarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{1+(2\pi n/a)^2} = \frac{a}{4}\left(\frac{e^a+1}{e^a-1}\right) - \frac{1}{2}.$$

b) Multiplying both sides of the above equation by $\frac{1}{a^2}$, we find

$$\sum_{n=1}^{\infty} \frac{1}{a^2+4\pi^2n^2} = \frac{1}{4a}\left(\frac{e^a+1}{e^a-1}\right) - \frac{1}{2a^2} = \frac{1}{4a}\,\frac{(1+a+\frac{a^2}{2}+\cdots)+1}{(\cancel{1}+a+\frac{a^2}{2}+\cdots)-\cancel{1}} - \frac{1}{2a^2} =$$

$$\frac{2+a+\frac{a^2}{2}+\cdots}{4a^2(1+\frac{a}{2}+\cdots)} - \frac{1}{2a^2} = \frac{(2+a+\frac{a^2}{2}+\cdots) - 2(1+\frac{a}{2}+\frac{a^2}{6}+\cdots)}{4a^2(1+\frac{a}{2}+\cdots)} = \frac{(\frac{1}{2}-\frac{1}{3})a^2+\cdots}{4a^2(1+\frac{a}{2}+\cdots)}$$

Thus, in the limit when $a \to 0$, we'll have:

$$\sum_{n=1}^{\infty} \frac{1}{4\pi^2n^2} = \frac{1}{24} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$