Problem 22) a) $F_L(s) = \int_{-\infty}^{\infty} f_L(x)\exp(-\mathrm{i}2\pi sx)\,dx$

$$= -\int_{-L}^{0}[1+(x/L)]\exp(-\mathrm{i}2\pi sx)\,dx + \int_{0}^{L}[1-(x/L)]\exp(-\mathrm{i}2\pi sx)\,dx$$

Change of variable: $x \to -x$ →

$$= -\int_{0}^{L}[1-(x/L)]\exp(+\mathrm{i}2\pi sx)\,dx + \int_{0}^{L}[1-(x/L)]\exp(-\mathrm{i}2\pi sx)\,dx$$

$$= -2\mathrm{i}\int_{0}^{L}[1-(x/L)]\sin(2\pi sx)\,dx$$

Integration by parts →

$$= 2\mathrm{i}\frac{[1-(x/L)]\cos(2\pi sx)}{2\pi s}\Big|_0^L + 2\mathrm{i}\int_0^L \frac{(1/L)\cos(2\pi sx)}{2\pi s}dx$$

$$= -\frac{2\mathrm{i}}{2\pi s} + 2\mathrm{i}\frac{\sin(2\pi sx)}{(2\pi s)^2 L}\Big|_0^L = \frac{1}{\mathrm{i}\pi s} - \frac{\sin(2\pi sL)}{\mathrm{i}2(\pi s)^2 L} = \frac{1-\mathrm{sinc}(2sL)}{\mathrm{i}\pi s}.$$

b) $\mathrm{Lim}_{L\to\infty} f_L(x) = \mathrm{Sgn}(x).$

c) Considering that $\mathrm{sinc}(2sL) \to 0$ when $L \to \infty$, we find: $\mathrm{Lim}_{L\to\infty} F_L(s) = 1/(\mathrm{i}\pi s)$.

Note: When $s = 0$, the limit of $\mathrm{sinc}(2sL) = \mathrm{sinc}(0) = 1$, independent of L. Consequently $\mathrm{Lim}_{L\to\infty} F_L(0) = 0$.

d)
$$f(x) = \int_{-\infty}^{\infty} F(s)\exp(\mathrm{i}2\pi xs)\,ds = \int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}2\pi xs)}{\mathrm{i}\pi s}ds.$$

Contours of integration in the complex plane are shown in the diagrams below. On the large semi-circle, the exponent of $\exp(\mathrm{i}2\pi xZ)$ must acquire a negative real part in order to satisfy Jordan's lemma. Thus, when x is positive (negative), the integration must be carried out in the upper (lower) half-plane. In each case, the contribution of the large semi-circle to the integral vanishes, and the integral over the real-axis becomes equal to that over the small semi-circle, taken counterclockwise when $x > 0$, and clockwise when $x < 0$.

$$f(x) = \int_{\substack{\text{small}\\ \text{semi-circle}}} \frac{\exp(\mathrm{i}2\pi xZ)}{\mathrm{i}\pi Z}dZ = \lim_{\varepsilon\to 0}\int_{\theta=0}^{\pm\pi}\frac{\exp[\mathrm{i}2\pi x\varepsilon\exp(\mathrm{i}\theta)]}{\mathrm{i}\pi\varepsilon\exp(\mathrm{i}\theta)}\mathrm{i}\varepsilon\exp(\mathrm{i}\theta)\,d\theta = \begin{cases}+\pi/\pi, & x>0;\\ -\pi/\pi, & x<0.\end{cases}$$

Clearly then $f(x) = \mathrm{Sgn}(x)$, as expected.