Problem 14) a) $\mathcal{F}\left\{\frac{1}{|x|^{1/2}}\right\} = \int_{-\infty}^{\infty} \frac{1}{|x|^{1/2}} e^{-i2\pi s x}\,dx =$

$$\int_{-\infty}^{0} \frac{1}{|x|^{1/2}} e^{-i2\pi s x}dx + \int_{0}^{\infty} \frac{1}{|x|^{1/2}} e^{-i2\pi s x}dx = \int_{0}^{\infty} \frac{1}{x^{1/2}} e^{+i2\pi s x}dx + \int_{0}^{\infty} \frac{1}{x^{1/2}} e^{-i2\pi s x}dx.$$

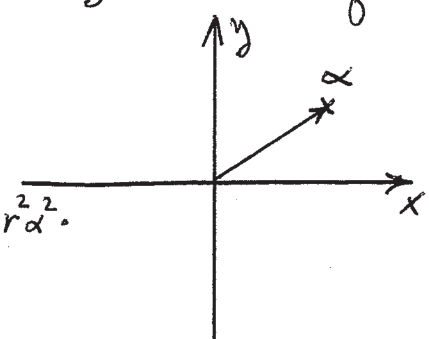
Change of variable $y^2 = x \Rightarrow 2y\,dy = dx \Rightarrow$

$$\mathcal{F}\left\{\frac{1}{|x|^{1/2}}\right\} = 2\int_0^\infty e^{i2\pi s y^2}dy + 2\int_0^\infty e^{-i2\pi s y^2}dy$$

The result is obviously an even function of s, because changing s to $-s$ will exchange the two integrals on the right-hand side of the above expression, but it does not change their sum. We thus write

$$\mathcal{F}\left\{\frac{1}{|x|^{1/2}}\right\} = 2\int_0^\infty e^{i2\pi|s|y^2}dy + 2\int_0^\infty e^{-i2\pi|s|y^2}dy.$$

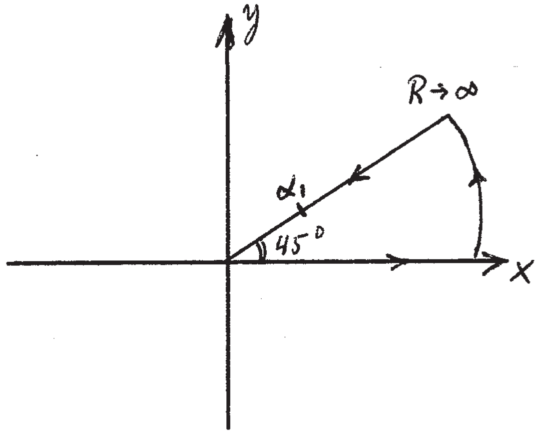
Choose a complex constant α in the z-plane, and let $z = r\alpha$ define a straight line from the origin to infinity along the direction of α. Then r, a real number, goes from 0 to ∞ as the point z moves from the origin to ∞ along the direction of α. On this line $i2\pi|s|z^2 = i2\pi|s|r^2\alpha^2$. Setting this equal to $\pm r^2$ yields



$$i2\pi|s|z^2 = i2\pi|s|r^2\alpha^2 = \pm r^2 \Rightarrow \alpha^2 = \frac{\pm 1}{i2\pi|s|} = \frac{\mp i}{2\pi|s|} = \frac{e^{\mp i\pi/2}}{2\pi|s|} \Rightarrow$$

$\alpha = \pm e^{\mp i\pi/4}/\sqrt{2\pi|s|}$. Of these four solutions for α we only need two, namely, $\alpha_1 = e^{+i\pi/4}/\sqrt{2\pi|s|}$, which is in the 1st quadrant, and $\alpha_2 = e^{-i\pi/4}/\sqrt{2\pi|s|}$, which is in the 4th quadrant.

For the first integral, $2\int_0^\infty e^{i2\pi|s|x^2}dx$, we use the following contour:

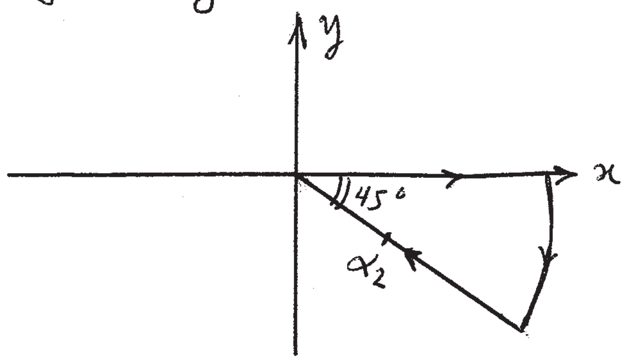


The function $e^{i2\pi|s|z^2}$ has no poles within this contour and its integral over the circular arc can be shown to approach zero as $R \to \infty$ (Problem 6.14).

Therefore, $2\int_0^\infty e^{i2\pi|s|x^2}dx = 2\int_{0\,(45^\circ\text{ line})}^{\infty} e^{i2\pi|s|z^2}dz = 2\alpha_1 \int_0^\infty e^{i2\pi|s|\frac{i}{2\pi|s|}r^2}dr$

$= \dfrac{2e^{i\pi/4}}{\sqrt{2\pi|s|}}\int_0^\infty e^{-r^2}dr = \dfrac{e^{i\pi/4}}{\sqrt{2|s|}}$ ← (see Chapter 1, Problem 4)

Similarly, to evaluate the second integral, $2\int_0^\infty e^{-i2\pi|s|x^2}dx$, we use the following contour:



The function $e^{-i2\pi|s|z^2}$ has no poles inside this contour. Also its integral over the circular arc can be shown to approach zero when $R \to \infty$ (Problem 6.14).

Therefore, $2\int_0^\infty e^{-i2\pi|s|x^2}dx = 2\int_{(-45^\circ\text{ line})} e^{-i2\pi|s|z^2}dz = 2\alpha_2\int_{r=0}^\infty e^{-i2\pi|s|\frac{-i}{2\pi|s|}r^2}dr$

$= \dfrac{2e^{-i\pi/4}}{\sqrt{2\pi|s|}}\int_0^\infty e^{-r^2}dr = \dfrac{e^{-i\pi/4}}{\sqrt{2|s|}}$ ← As before, $\int_0^\infty e^{-r^2}dr = \frac{\sqrt{\pi}}{2}$.

Consequently, $\mathcal{F}\left\{\dfrac{1}{|x|^{1/2}}\right\} = \dfrac{e^{i\pi/4}+e^{-i\pi/4}}{\sqrt{2|s|}} = \dfrac{2\cos\pi/4}{\sqrt{2|s|}} = \dfrac{1}{\sqrt{|s|}} = \dfrac{1}{|s|^{1/2}}$

b) $$\mathcal{F}\left\{\frac{\text{sign}(x)}{|x|^{1/2}}\right\} = \int_{-\infty}^{0} \frac{-1}{|x|^{1/2}} e^{-i2\pi sx}\,dx + \int_{0}^{\infty} \frac{+1}{|x|^{1/2}} e^{-i2\pi sx}\,dx$$

$$= \int_{0}^{\infty} \frac{1}{x^{1/2}} e^{-i2\pi sx}\,dx - \int_{0}^{\infty} \frac{1}{x^{1/2}} e^{+i2\pi sx}\,dx$$

This is an odd function of s, so changing the sign of s will reverse the sign of the Fourier Transform. We thus write

$$\mathcal{F}\left\{\frac{\text{Sign}(x)}{|x|^{1/2}}\right\} = 2\,\text{Sign}(s)\left\{\int_{0}^{\infty} e^{-i2\pi|s|y^2}\,dy - \int_{0}^{\infty} e^{i2\pi|s|y^2}\,dy\right\}$$

Using the results of part (a), we'll have:

$$\mathcal{F}\left\{\frac{\text{Sign}(x)}{|x|^{1/2}}\right\} = \text{Sign}(s)\left\{\frac{e^{-i\pi/4}}{\sqrt{2|s|}} - \frac{e^{+i\pi/4}}{\sqrt{2|s|}}\right\} = \text{Sign}(s)\,\frac{-2i\sin(\pi/4)}{\sqrt{2|s|}}$$

$$\Rightarrow \mathcal{F}\left\{\frac{\text{Sign}(x)}{|x|^{1/2}}\right\} = -i\,\text{Sign}(s)/\sqrt{|s|}\,.$$

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