Problem 12) We begin by finding the Fourier transform of $\exp(\mathrm{i}ax^2)$, then use its real and imaginary parts to determine the Fourier transforms of the functions $\cos(ax^2)$ and $\sin(ax^2)$. This is possible because both $\cos(ax^2)$ and $\sin(ax^2)$ are real <u>and</u> even functions of x and, therefore, their Fourier transforms must be real as well.

$$\mathcal{F}\{e^{\mathrm{i}ax^2}\} = \int_{-\infty}^{\infty} e^{\mathrm{i}ax^2} e^{-\mathrm{i}2\pi sx}\mathrm{d}x = \int_{-\infty}^{\infty} e^{\mathrm{i}a[x-(\pi s/a)]^2-\mathrm{i}(\pi^2 s^2/a)}\mathrm{d}x \quad \leftarrow \text{completing the square}$$

$$= e^{-\mathrm{i}(\pi^2 s^2/a)}\int_{-\infty}^{\infty} e^{\mathrm{i}a[x-(\pi s/a)]^2}\mathrm{d}x = e^{-\mathrm{i}(\pi^2 s^2/a)}\int_{-\infty}^{\infty}\exp(\mathrm{i}ay^2)\,\mathrm{d}y \quad \leftarrow \text{change of variable: } y = x-(\pi s/a)$$

$$\text{change of variable: } x=\sqrt{a}y \rightarrow = \frac{1}{\sqrt{a}}e^{-\mathrm{i}(\pi^2 s^2/a)}\int_{-\infty}^{\infty}\exp(\mathrm{i}x^2)\,\mathrm{d}x = \frac{2}{\sqrt{a}}e^{-\mathrm{i}(\pi^2 s^2/a)}\left(\int_0^{\infty}\cos x^2\,\mathrm{d}x + \mathrm{i}\int_0^{\infty}\sin x^2\,\mathrm{d}x\right)$$

$$= \frac{2}{\sqrt{a}}e^{-\mathrm{i}(\pi^2 s^2/a)}\left(\frac{\sqrt{\pi}}{2\sqrt{2}} + \mathrm{i}\frac{\sqrt{\pi}}{2\sqrt{2}}\right) = \sqrt{\pi/a}\,e^{-\mathrm{i}(\pi^2 s^2/a)}e^{\mathrm{i}\pi/4}$$

$$= \sqrt{\pi/a}\left[\cos\left(\frac{\pi^2 s^2}{a}-\frac{\pi}{4}\right) - \mathrm{i}\sin\left(\frac{\pi^2 s^2}{a}-\frac{\pi}{4}\right)\right].$$

Consequently,

$$\mathcal{F}\{\cos(ax^2)\} = \sqrt{\pi/a}\cos\left(\frac{\pi^2 s^2}{a}-\frac{\pi}{4}\right),$$

$$\mathcal{F}\{\sin(ax^2)\} = \sqrt{\pi/a}\cos\left(\frac{\pi^2 s^2}{a}+\frac{\pi}{4}\right).$$