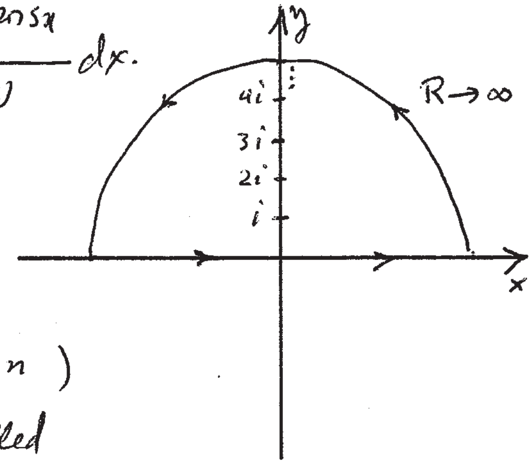
Problem 11) $\mathcal{F}\left\{\frac{2x}{\sinh(\pi x)}\right\} = \int_{-\infty}^{\infty} \frac{2x e^{-i2\pi s x}}{\sinh(\pi x)} dx.$



The poles of the integrand are the zeros of $\sinh(\pi z)$, namely, $e^{\pi z} - e^{-\pi z} = 0 \Rightarrow$

$e^{2\pi z} = 1 \Rightarrow e^{2\pi x} e^{2\pi i y} = e^{i2n\pi} \Rightarrow (x=0,\ y=n)$

$\Rightarrow z_n = in$. The pole at $z_0 = 0$ is cancelled by the zero in the numerator, so the remaining poles of the integrand are at $z_1 = i$, $z_2 = 2i$, $z_3 = 3i, \ldots$. All these are simple poles.

The integral on the large circle goes to zero as $R \to \infty$ (Jordan's lemma).

At each pole we have $\sinh(\pi z) = \sinh(\pi z_n) + \pi \cosh(\pi z_n)(z - z_n) + \cdots$

$$\simeq 0 + \pi \cosh(i\pi n)(z - in) = \pi \frac{e^{i\pi n} + e^{-i\pi n}}{2}(z - in) = \pi(-1)^n (z - in).$$

The residue at z_n is thus given by $\frac{2 z_n e^{-i2\pi s z_n}}{\pi(-1)^n} = \frac{2in}{\pi}(-1)^n e^{2\pi n s}$

Cauchy's theorem then yields $\int_{-\infty}^{\infty} \frac{2x}{\sinh(\pi x)} e^{-i2\pi s x} dx = 2\pi i \sum_{n=1}^{\infty} \frac{2in}{\pi}(-1)^n e^{2\pi n s}$

$$\Rightarrow \mathcal{F}\left\{\frac{2x}{\sinh(\pi x)}\right\} = -4 \sum_{n=1}^{\infty} n(-e^{2\pi s})^n.$$

Now, $\sum_{n=1}^{\infty} (-e^{2\pi s})^n = \frac{-e^{2\pi s}}{1+e^{2\pi s}}$ (Geometric Series).

$$\frac{d}{ds}\sum_{n=1}^{\infty}(-1)^n e^{2\pi n s} = \sum_{n=1}^{\infty}(-1)^n 2\pi n e^{2\pi n s} = 2\pi \sum_{n=1}^{\infty} n(-e^{2\pi s})^n \Rightarrow$$

$$\sum_{n=1}^{\infty} n(-e^{2\pi s})^n = \frac{1}{2\pi}\frac{d}{ds}\left(\frac{-e^{2\pi s}}{1+e^{2\pi s}}\right) = -\frac{1}{2\pi}\frac{2\pi e^{2\pi s}(1+e^{2\pi s}) - 2\pi e^{2\pi s} e^{2\pi s}}{(1+e^{2\pi s})^2}$$

$$= -\frac{e^{2\pi s}}{(1+e^{2\pi s})^2} = -\frac{1}{(e^{-\pi s}+e^{\pi s})^2} = -\frac{1/4}{\cosh^2(\pi s)}$$

Therefore, $\mathcal{F}\left\{\frac{2x}{\sinh(\pi x)}\right\} = \frac{1}{\cosh^2(\pi s)}$. ✓